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On the Critical Behavior of the Magnetization in High-Dimensional Ising Models[1]

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We derive rigorously general results on the critical behavior of the magnetization in Ising models, as a function of the temperature and the external field. For the nearest-neighbor models it is shown that in $d \geqslant 4$ dimensions the magnetization is continuous at T_c and its critical exponents take the classical values $\delta = 3$ and $\beta = \frac{1}{2}$, with possible logarithmic corrections at $d = 4$. The continuity, and other explicit bounds, formally extend to $d > 3\frac{1}{2}$. Other systems to which the results apply include long-range models in $d = 1$ dimension, with $1/|x-y|^{\lambda}$ couplings, for which $2/(\lambda - 1)$ replaces d in the above summary. The results are obtained by means of differential inequalities derived here using the random current representation, which is discussed in detail for the case of a nonvanishing magnetic field.

KEY WORDS: Critical exponents; spontaneous magnetization; Ising model; upper critical dimension; random-current representation.

1. INTRODUCTION

1.1. The Main Results

The critical behavior of the order parameter in statistical mechanical systems is characterized by a number of critical exponents. A great deal of information may be deduced, or rather guessed in a fairly sophisticated manner, about such quantities by using the phenomenological scaling theory, and a field theoretical formulation of the critical behavior.

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However, due to the interest both in these "universal quantities" and in the underpinning of the general methods quated above, it is still very desirable to study the critical behavior by methods which deal with these issues within the given model, i.e., assuming only the "first principles" of statistical mechanics.

In this paper we consider, from this point of view, the critical behavior of the magnetization in translation-invariant ferromagnetic Ising spin systems with the Hamiltonian

$$H = -\frac{1}{2}\sum_{x,y} J_{xy}\sigma_x\sigma_y - \sum_x h_x\sigma_x \tag{1.1}$$

Our main purpose is to analyze the *nonsymmetric* regime, in the vicinity of the critical point $(T = T_c, h = 0)$.

Many aspects of the phase transition in such models have by now been derived by rigorous methods. In particular, the notion of the upper critical dimension has been confirmed by proofs of the facts that in the nearest-neighbor models in $d \underset{(-)}{>} 4$ dimensions the critical exponents α_+, γ_+ of the specific heat[1] and of the magnetic susceptibility[2,3] take their "classical" values, and the scaling limits are gaussian.[2,4]

It should, however, be emphasized that the above results refer to the critical behavior in the very special regime $R_0 = \{T > T_c, h = 0\}$ in which the symmetry $\sigma \to -\sigma$ is not removed by either an explicit term in the Hamiltonian or by dynamical symmetry breaking. The analysis has not been extended to the nonsymmetric regime $h \neq 0$ or $T < T_c$ where other issues (see Ref. 5) have to be confronted.

In this work we address one of these pending issues: the upper critical dimension for the magnetization's critical exponents β and δ. This question may, in fact, be viewed as symptomatic of the more general issue of the extent of the "triviality" picture for $d > 4$ dimensions, which was rigorously derived for the critical behavior of the Ising model only in the symmetric "high-temperature" phase, i.e., the region R_0.

The critical exponents mentioned above are defined by the limiting values of the logarithmic ratios (generally presumed to exist) in the following expressions:

$$\begin{aligned} M(T, h=0) &\propto (T_c - T)^{\beta} && \text{for} \quad T \leqslant T_c \\ M(T=T_c, h) &\propto h^{1/\delta} && \text{for} \quad h \geqslant 0 \end{aligned} \tag{1.2}$$

Our results combined with previous results quoted below show that for the translation-invariant nearest neighbor models in $d \geqslant 4$ dimensions these limits exist and the critical exponents take their mean field values:

$$\beta = 1/2 \qquad \text{and} \qquad \delta = 3 \tag{1.3}$$

As was the case with other results about critical exponents for $d \geqslant 4$ (i.e., the uniform boundedness of the specific heat in the high-temperature phase[1] for $d > 4$, and the equality[2,3] $\gamma_+ = 1$), the condition of high dimension enters here only through the requirement of the finiteness of the "bubble diagram":

$$B_0 = \sum_x \langle \sigma_0 \sigma_x \rangle_{h=0}^2 < \infty \tag{1.4}$$

at $T = T_c$. Thus, the values (1.3) extend also to the other Ising systems for which (1.4) holds. An example would be the one-dimensional ferromagnetic models with $J_{xy} = |x - y|^{-\lambda}$ for $1 < \lambda < 1.5$.[6]

Before presenting the results more explicitly, let us make some comments about the notation which is followed. The relevant parameters for our work are chosen to be β and βh. Thus, we denote by $\partial/\partial\beta$ the derivative performed at constant βh and define

$$\chi \equiv \partial M/\partial(\beta h) = \sum_x \langle \sigma_0, \sigma_x \rangle = \sum_x \langle \sigma_0 \sigma_x \rangle - \langle \sigma_0 \rangle \langle \sigma_x \rangle \tag{1.5}$$

For the analysis of the critical regime it is convenient also to use the parameter $t = \beta_c - \beta$ which, with our choice, is positive on the high-temperature side of the critical point.

We shall assume in this paper that the system under discussion exhibits a sharp transition, at $h = 0$, from the high-temperature regime, for which $\chi < \infty$, to the low-temperature regime which is characterized by the long-range order. Let us note that the coincidence of the two notions of a critical point T_c induced by these conditions was proven in Ref. 6 for a class of Ising models which satisfy a certain regularity condition. This condition was proven there to be satisfied by the n.n. model in dimensions $d > 2$ and the one-dimensional long-range systems for $1 < \lambda < 2$.

It was also proven in Ref. 6 that under the above-mentioned regularity condition

$$M(t, h=0) \geqslant c\,|t|^{1/2} \qquad \text{for} \quad t < 0 \tag{1.6}$$

(with a constant which is remarkably independent of the bound which establishes the regularity). We shall assume here that such a bound is satisfied in the models we discuss. It seems to us (although we have at present no proof)[4] that the last two assumptions are valid in any dimension for any translation-invariant ferromagnetic system with $\sum_x J_{0x} < \infty$.

The relation (1.6) may be regarded as a mean-field type bound on the critical behavior of $M(t, h=0)$. Another generally valid relation of that sort is

$$M(t=0, h) \geqslant c h^{1/3} \tag{1.7}$$

[4] This claim has now been proved for all translation invariant models (M. Aizenman, D. Barsky and R. Fernández; in preparation).

The lower bound (1.7) was derived by Fröhlich and Sokal,[7] and independently by C. Newman,[8] who pointed out that it follows from (1.6) by an argument involving the GHS inequality (this observation is explained and further expanded here in Sec. 2 and Appendix B).

The main goals of this work are to provide *upper* bounds on M which are complementary to (1.6) and (1.7). To state the results, let us distinguish among three kinds of critical behavior of the bubble as $t \downarrow 0^+$:

$$\begin{aligned} &\text{Case 1: } B_0 \leqslant c \\ &\text{Case 2: } B_0 \leqslant c\,|\ln t|^{\omega} \\ &\text{Case 3: } B_0 \leqslant c t^{-\zeta} \end{aligned} \tag{1.8}$$

The results derived here, together with (1.6) and (1.7), show that in the critical regime the magnetization obeys the following bounds:

(i) At $h = 0$, $t \uparrow 0^-$:

$$c\,|t|^{1/2} \leqslant M \leqslant \begin{cases} c_1\,|t|^{1/2} & \text{Case 1} \\ c_2\,|t|^{1/2}\,|\ln |t||^{3\omega/2} & \text{Case 2} \\ c_3\,|t|^{(1-3\zeta)/2} & \text{Case 3 with } \zeta < 1/3 \end{cases} \tag{1.9}$$

(ii) Along *any* ray $t = a\beta h$, $h \downarrow 0^+$ we have the *asymptotic* inequalities:

$$c(\beta h)^{1/3} \leqslant M \leqslant \begin{cases} c_1(\beta h)^{1/3} & \text{Case 1} \\ c_2(\beta h)^{1/3}\,|\ln(\beta h)|^{\omega} & \text{Case 2} \\ c_3(\beta h)^{(1-3\zeta)/(3-3\zeta)} & \text{Case 3 with } \zeta < 1/3 \end{cases} \tag{1.10}$$

with constants which are asymptotically independent of the slope a of the ray.

It may be worth emphasizing the following explicit implication of the above bounds:

(iii) For cases 1, 2, and 3 with $\zeta < 1/3$ the magnetization is continuous at T_c.

The above bounds imply the following inequalities for the critical exponents β and δ, defined as limits of logarithmic ratios:

$$\frac{1}{2} \geqslant \beta \geqslant \begin{cases} \dfrac{1}{2} & \text{Cases 1 and 2} \\[2ex] \dfrac{1-3\zeta}{2} & \text{Case 3 with } \zeta < 1/3 \end{cases} \tag{1.11}$$

$$3 \leqslant \delta \leqslant \begin{cases} 3 & \text{Cases 1 and 2} \\[1ex] 3\left(\dfrac{1-\zeta}{1-3\zeta}\right) & \text{Case 3 with } \zeta < 1/3 \end{cases} \tag{1.12}$$

The results described above apply in particular to the translation-invariant n.n. models for which the known bounds of Ref. 9, 10, and 3 can be combined to prove the following dimension dependence (see Appendix A):

$$\begin{array}{rl} d>4 & \text{Case 1} \\ d=4 & \text{Case 2, with } \omega=1 \\ 2<d<4 & \text{Case 3, with } \zeta=(4-d)/(d-2) \end{array} \tag{1.13}$$

However, since the restriction $\zeta < 1/3$ is satisfied only for $d > 3\frac{1}{2}$, and our analysis was developed only for lattices of integral dimension, we have no meaningful new results for the n.n. models in $d < 4$ dimensions. Nevertheless, the results for Case 3 are meaningful in the context of the next example.

For the one-dimensional long-range models with $J_{xy} = |x-y|^{-\lambda}$, we have

$$\begin{array}{rl} 1<\lambda<1\frac{1}{2} & \text{Case 1} \\ \lambda=1\frac{1}{2} & \text{Case 2, with } \omega=1 \\ 1\frac{1}{2}<\lambda<2 & \text{Case 3, with } \zeta=(2\lambda-3)/(2-\lambda) \end{array} \tag{1.14}$$

Expanding on the observation of Newman mentioned after (1.7), we also obtain here extrapolation principles relating not only β with δ but, more generally, the critical behavior along any two different (straight) lines of the approach to the critical point in the $(\beta, \beta h)$ plane (with the exception of the line of the symmetric regime $R_0 = \{T > T_c, h = 0\}$). This interesting, and useful, implication of these principles (Lemma 2.2) provides a confirmation of one of the basic predictions of the scaling theory.

1.2. Remarks on the Method

The method used to obtain the above bounds on the critical behavior falls within the general approach in which one searches for, and then applies, differential inequalities relating the few relevant physical quantities. The difficulty is, of course, that we are interested in understanding the non-perturbative regime of a system with infinitely many degrees of freedom. The method introduced in Ref. 2 is to consider a diagrammatic representation, which often is taken as the basis for a high-temperature expansion, as a statistical mechanical system endowed with a probability measure, in which various physical quantities are given a stochastic geometric interpretation. The geometric representations are often quite suggestive of non-perturbative relations, some of which may even be proven to be true.

The geometric picture may become yet more transparent upon a partial resummation of the diagrammatic representation, in which one sums over all the terms with the same "backbone." The result is an expansion whose terms can be identified with simple geometric objects, e.g., random walks (with certain interactions.) Different variants of random walk expansions of this kind appear in Refs. 2, 3, and 11.

The latter representations for the correlation functions of the Ising model share some of the basic properties with the expansion developed in the works of Brydges, Fröhlich, and Spencer[12] and Fröhlich[4] (see also Ref. 13), from the "Symanzik representation" of a Euclidean field theory. This point will be further emphasized in another paper, co-authored jointly with Fröhlich and Sokal,[14] in which we shall address also some other aspects of the "triviality" issue for the nonsymmetric approach to the critical point in dimensions $d > 4$.

For the purpose of this work, it was instructive to have a random walk expansion for the magnetization in the presence of an external field. This concept first appeared in the work of Fröhlich and Sokal,[7] and we would like to thank them for conveying to us their very stimulating ideas prior to publication, and many useful comments. The close analogy between the two expansions for the order parameter found in Ref. 7 and in this paper will be explicity discussed in Ref. 14, and, in fact, these three papers may be regarded as one unit.

Alas, the general properties of the (simpler) random walk expansions did not suffice for the derivation of the relations on which our results are based. In fact, our analysis required the full power of the "random current" representation,[2] which we discuss in Sec. 3. When applicable, "random walk" expressions have a simpler form, and we have attempted therefore to use them whenever possible. This formalism is introduced in Sec. 4.

The main new ingredient which was needed for the derivation of (1.9) and (1.10) was the lower bound in the following result (Theorem 5.7 below). For a translation-invariant Ising model, on Z^d

$$\frac{|1-Q\bar{B}_0|_+^2}{96 B_0(1+2\bar{B}_0)^2} \tanh(\beta h)\,\chi^4 \leqslant \left|\frac{\partial\chi}{\partial(\beta h)}\right| \leqslant 32\beta\,|J| \left[1+\frac{1}{2}Q\right] \frac{M^4}{(\tanh(\beta h))^3} \tag{1.15}$$

where $|x|_+ = \max(x, 0)$, $\bar{B}_0 = B_0\,|J|\,\beta$, $Q(t, h) \equiv \tanh(\beta h)/M\,|J|\,\beta \leqslant (\beta\,|J|\,\chi)^{-1}$ (which is vanishingly small in the vicinity of the critical point), and in the upper bound $h \neq 0$.

It should be emphasized that the quantity B_0 [defined in (1.4)] refers to the untruncated bubble diagram at $h = 0$, and the given value of β. Thus $B_0(\beta) = \infty$ for $\beta > \beta_c$, and therefore the lower bound in (1.15) provides direct information only about the region $\{\beta \leqslant \beta_c, h \geqslant 0\}$. However, the extrapolation principles mentioned at the end of the previous subsection allow us to extend the consequences of (1.15) beyond that domain.

In the process of proving (1.15) we also obtain and use the lower bound of the following inequality which may be of independent interest. With the same hypothesis as for (1.15):

$$\frac{|1 - Q\bar{B}_0|_+}{1 + 2\bar{B}_0} |J|\, M\chi \leqslant \frac{\partial M}{\partial \beta} \leqslant |J|\, M\chi \tag{1.16}$$

The lower bound is proven in Theorem 5.6 below, and the upper bound is a consequence[15] of the GHS inequality.[16]

To a certain extent, a precursor for the above inequalities may be found in a result of Ref. 3, on $\partial\chi/\partial\beta$ in the symmetric regime R_0 (these results formed also the basis for Ref. 6 and are further discussed there). A common feature is that one has two bounds: the first resembling the mean field approximation, and the second (in the opposite direction) incorporating "loop corrections". Since the quantity B_0 is not infinitesimal, and in fact is divergent at T_c below the upper critical dimension, corrective factors of the form $(1 - \text{const.}\, B_0)$ are of much more limited use than factors like $1/(1 + \text{const.}\, B_0)$ which can be seen in (1.15) and (1.16). The latter imply, of course, information on the corrections to "all orders in B_0."

A significant difference between the nonsymmetric regime and R_0 is that one has to contend with cancellation effects [as in (1.15)], which raise the level of difficulty. Furthermore, the analysis of the inequalities does not proceed directly along the lines of interest (like the critical isotherm $\{T = T_c\}$), requiring, in addition to (1.15), some extrapolation arguments. We shall present those in the next section before proceeding to the more technical part of this work, which is the derivation of (1.15).

2. FROM DIFFERENTIAL INEQUALITIES TO CRITICAL EXPONENTS

In this section we shall discuss the derivation of the main results (1.9)–(1.10) from the differential inequality (1.15). Let us first remark that not all inequalities of this type can be integrated. Furthermore, even when a direct integration is possible, the result may not be entirely satisfactory. We shall overcome these difficulties with the help of an extrapolation principle [and (1.6)]. It should, of course, be noted that if one is willing to *assume* a strict power law behavior in (1.2), then (1.15) does immediately lead to the critical exponents given in (1.10) for Case 1, where the bubble diagram is uniformly bounded.

2.1. Integration of the Lower Bound in (1.15)

First, we need to determine the shape of the region $R_1 = \{(\beta, \beta h) \mid Q\bar{B}_0 < 1\}$ where the lower bound in (1.15) is nontrivial. For this, some information is needed about the behavior of $B_0(t)$, which is also relevant for the bound itself, and of $M(t, h)$. Under the assumptions made in the introduction, R_1 contains the region where $t > 0$ and

$$\beta h \leqslant \begin{cases} c & \text{Case 1} \\ c\,|\ln t|^{-3\omega/2} & \text{Case 2} \\ ct^{3\zeta/2} & \text{Case 3} \end{cases} \tag{2.1}$$

(We used the lower bound in M given in (1.10) which is derived independently below.) In the first two cases, and the third one as long as $\zeta < \frac{2}{3}$ ("$d > 3\frac{1}{5}$,") the regime described by (2.1) is convex and includes a final segment of the approach to the critical point along any of the rays $t = a\beta h$ with $a > 0$ (see Fig. 1)

The lower bound in (1.15), the mean-field bound $\beta_J |J| \geqslant 1$, and the GHS inequality imply that in a region of the form (2.1)

$$\frac{\partial \chi}{\partial(\beta h)} \leqslant -\frac{\bar{c}\beta h \chi^4}{B_0^3} \tag{2.2}$$

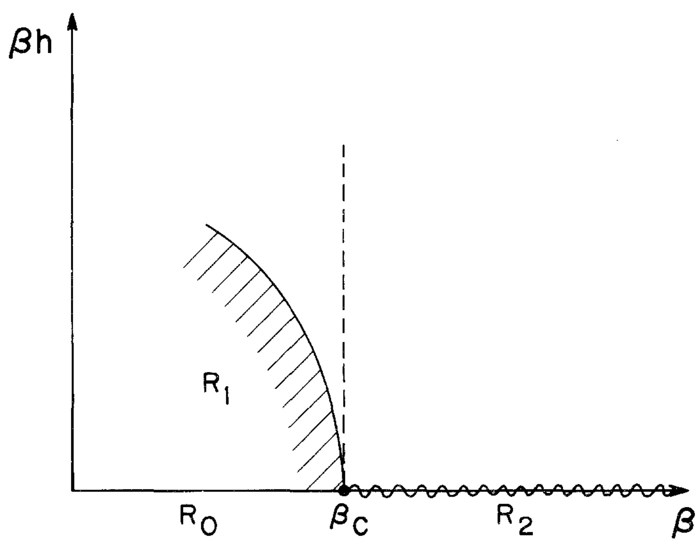


Fig. 1. Regimes involved in the process of extrapolation.

with

$$\bar{c} = \frac{0.9}{9.96(\beta_c\,|J|)^2} \tag{2.3}$$

Integrating (2.2) "up" from $\beta h = 0$ along a line $t = \text{const} > 0$ we get

$$\frac{1}{\chi^3} - \frac{1}{\chi_0^3} \geq (3\bar{c}/2)\frac{(\beta h)^2}{B_0^3} \Rightarrow \chi \leq (2/(3\bar{c}))^{1/3}\frac{B_0}{(\beta h)^{2/3}}$$

A further integration yields

$$M \leq cB_0(\beta)\ (\beta h)^{1/3} \tag{2.4}$$

with $c = 3(2/(3\bar{c}))^{1/3}$. Here we have used the boundary value for $t > 0$ $M(t, h = 0) = 0$.

It should be pointed out that the above integration could be performed because at each step the bound was reduced to a first-order differential inequality. Since the upper bound of (1.15) is not of that form, we are not able to use it to obtain an inequality which should supplement (2.4).

The bound (2.4) is still rather unsatisfactory for two reasons:

(i) It does not apply to the critical isotherm in Cases 2 and 3, and in particular to the nearest-neighbor Ising model in $d = 4$ dimensions.

(ii) From a general perspective, the form of (2.4) is totally unnatural when viewed from the vantage point of the scaling theory,[17] which predicts the same critical behavior along any ray $t = a\beta h$, $h \geq 0$.

The three limitations pointed out above will be removed by an application of an extrapolation technique which will be introduced next.

2.2. Extrapolation Principles

Our procedure is an elaboration on the basic observation of Newman[8] that the GHS inequality can be used to relate the critical behavior along different lines of approach of the critical point. (We learned from J. Lebowitz that somewhat weaker information can be deduced from the FKG inequalities.[18]) A particularly useful way to state the basic principle is to note that the GHS inequality has the consequence that the slope of any constant magnetization line in the $(t, \beta h)$ plane satisfies

$$-M\,|J| \leq \left.\frac{\partial(\beta h)}{\partial t}\right|_{M=\text{const}} \quad (\leq 0)$$

This remark leads to the following extrapolation principles (derived in Appendix B).

Lemma 2.1. Translation invariance, GHS, and Griffiths (II)[19] inequalities imply that for every point $(\beta_0, \beta_0 h_0)$, with magnetization $M_0 = M(\beta_0, \beta_0 h_0)$, and $\Delta > 0$:

$$M(\beta_0 + \Delta, \beta_0 h_0 + M_0 \, |J| \, \Delta) \geqslant M(\beta_0, \beta_0 h_0) \tag{2.5}$$

Lemma 2.2. If along a ray $t = a\beta h$, $h \geqslant 0$

$$c_1(\beta h)^{\alpha_1} \, |\ln (\beta h)|^{\omega_1} \, (1 + O(\beta h)) \leqslant M \leqslant c_2(\beta h)^{\alpha_2} \, |\ln(\beta h)|^{\omega_2} \, (1 + O(\beta h)) \tag{2.6}$$

with $0 < \alpha_i < 1$ and $\omega_i \geqslant 0$, then the same inequality (with the same c_i, α_i, ω_i) is asymptotically true for any other ray $t = b\beta h$, $h \geqslant 0$.

Lemma 2.2 shows that the behavior along the critical isotherm is the same as for any other ray $t = a\beta h$. The next result relates those rays with R_2, the multiphase regime.

Lemma 2.3.

1. If along a ray $t = a\beta h$, $h \geqslant 0$

$$M \leqslant c(\beta h)^{\alpha} \, |\ln(\beta h)|^{\omega} \, (1 + O(\beta h))$$

with $0 < \alpha < 1$ and $\omega \geqslant 0$, then in the region $R_2 = \{(\beta, \beta h = 0) \mid \beta > \beta_c\}$

$$M \leqslant (|J| \, c^{1/\alpha})^{\alpha/(1-\alpha)} \, |t|^{\alpha/(1-\alpha)} \, |\ln(M \, |t|)|^{\omega/(1-\alpha)} (1 + O(|t|) \tag{2.7}$$

2. If in region R_2

$$M \geqslant c \, |t|^{\lambda} \, (1 + O(t)) \tag{2.8}$$

with $\lambda \geqslant 0$, then, along any ray $t = a\beta h$, $h \geqslant 0$,

$$M \geqslant (c \, |J|^{-\lambda})^{\lambda/(1+\lambda)} (\beta h)^{\lambda|(1+\lambda)} (1 + O(\beta h)) \tag{2.9}$$

2.3. Derivation of the Main Results

Having stated the extrapolation principles, we may now describe the flow diagram for the derivation of the bounds (1.9)–(1.10) In essence, for the upper bounds on M we will use (2.4) along some ray, and then extrapolate it to all other rays and to the region R_2 using Lemma 2.2 and Lemma 2.3 part 1. For the lower bounds the direction of extrapolation is the opposite, namely from R_2 to the rays.

The argument goes as follows. We start from the lower bound (1.6) for the magnetization in the multicritical regime. Using (2.9) it implies the lower bound in (1.10). To prove the upper bounds we first make use of the latter result to obtain the description (2.1) of R_1 and thus to conclude that there exists a ray—in fact any ray $t = a\beta h$ with $a > 0$ will do—for which the bound (2.4) holds. For Cases 1 and 2 the desired upper bounds follow readily: the one in (1.10) is just an application of (2.4) along any of such rays, and then the extrapolation principle (2.7) yields the one in (1.9). For this last step in Case 2 we eliminate the reference to $\ln |M|$ in the RHS of (2.7) by invoking the already proven lower bound in (1.9).

For an upper bound for Case 3 one can proceed in the same way, namely insert the lower line of (1.8) in (2.4) for $t = a\beta h$ $(a > 0)$. One gets $M \leqslant \text{const}\, h^{(1/3) - \zeta}$ which, of course, is of interest only if $\zeta < \frac{1}{3}$. In the following argument we shall improve this bound and get a better exponent for the RHS. We note that the (unnatural) upper bound (2.4) has a different behavior with respect to t than to h. In fact, the RHS increases with h but decreases with t. Thus, there is still room for improvement, which can be obtained by an optimization procedure based on (2.5). Indeed, we have from (2.5), (2.4), and the lowest line in (1.8) that, for any $\Delta \geqslant 0$,

$$M(t, \beta h) \leqslant c(\beta h + M(t, \beta h)\, |J|\, \Delta)^{1/3}\, (t + \Delta)^{-\zeta} \tag{2.10}$$

provided the point $(t', (\beta h)') = (t + \Delta, \beta h + M(t, \beta h)\, |J|\, \Delta)$ is in the region R_1. We shall use this inequality with the following value of Δ:

$$\Delta = \frac{3\zeta \beta h}{|J|\, M(t, \beta h)(1 - 3\zeta)} \tag{2.11}$$

(The optimal value of Δ for (2.10) has an additional term $(-M\, |J|\, t)$ in the numerator. However, its incorporation would not lead to any significant improvement.) Again, our analysis can proceed only if $\zeta < 1/3$. For the points along the critical isotherm the above procedure yields

$$M(0, \beta_c) \leqslant c_d((\beta_c h)')^{1/3}\, (t')^{-\zeta} \tag{2.12}$$

with

$$(\beta_c h)' = \frac{\beta_c h}{1 - 3\zeta} \tag{2.13}$$

$$t' = \frac{3\zeta}{M(0, \beta_c h)\, |J|}\, (\beta_c h)' \tag{2.14}$$

It is not hard to see that for $\zeta < 1/3$ the curve $\beta_c h \to (t'(\beta_c h), (\beta_c h)'\, (\beta_c h))$ is asymptotically inside R_1 so the bound (2.12) is applied properly.

Writing (2.12) in an explicit form, we see that the optimization produced the improved power:

$$M(0, \beta_c) \leqslant \left[\frac{c\,|J|^{\zeta}}{(1-3\zeta)^{1/3}\,(3\zeta)^{\zeta}}\right]^{1/(1-\zeta)} (\beta_c h)^{(1/3-\zeta)/(1-\zeta)} \tag{2.15}$$

By Lemma 2.2 such a bound is asymptotically satisfied also along any other ray $t = a\beta h$. (In fact, the above analysis may be applied directly to such cases, even with $a < 0$, yielding the same result.)

Inequality (2.15) is the upper bound in (1.10) for Case 3; the upper bound in (1.9) follows by (2.7).

3. THE RANDOM CURRENT REPRESENTATION

3.1. General Features

In this section we introduce the random current representation (RCR), which is the basic formalism used to prove the inequalities (1.9) and (1.10). Our goal here is to define it, describe some of its most useful properties, and demonstrate the basic techniques by performing the delicate cancellations which occur in some of the truncated correlation functions which are relevant for our discussion.

The starting point of the RCR is similar to that of the high-temperature expansions (HTE). It may therefore be helpful to emphasize in what the use of the RCR, as developed in Ref. 2, differs from the more standard techniques.

Both RCR and HTE start from diagrammatic representations for the free energy and the correlation functions. An approach which is often followed for the development of a HTE is to extract from a diagrammatic representation an expansion in more primitive terms, which are expressed by diagrams with a higher degree of connectivity (connected, 1-particle irreducible, etc.). One may often obtain by such methods an expansion which converges in the high-temperature regime, and which may be used for various purposes like the construction and the study of the infinite volume limit.

Basic points at which the approach which led to RCR departs from the above scenario are:

(i) Probabilistic interpretation. Instead of attempting to sum the diagrammatic representation, one looks at its terms as defining a new (interacting) statistical-mechanical system, and poses questions about the properties of the "typical" (i.e., dominant) terms. There is never a question of convergence since the setting is basically probabilistic, although there are problems related to the uniqueness of the infinite volume limit and its independence of the boundary conditions. We shall avoid such issues by basing our analysis on finite systems.

(ii) Nonperturbative geometric identification. For a number of observables there is a clear identification of their expectation values with the probabilities of precisely described geometric events in the ensemble of diagrams mentioned above. For the systems discussed here, the diagrams can be viewed as describing random currents. A basic example of an event related to a physical quantity is the event that the currents connect a specified set of sites. This may be the point to explain that the purpose of some of the seemingly cumbersome operations, like the use of duplicated ensembles, is to arrive at precise geometric characterizations in terms of events of this kind. In particular, this is required for the performance of the rather delicate cancellations necessary for the interpretation of the truncated correlation functions. A special attribute of the RCR is the existence of such identities at the nonperturbative level.

(iii) Inequalities. An attractive feature of geometric representations is that they provide a "robust" description of subtle correlation effects, and suggest important relations which may be expressed by inequalities. The "lexicon" described above may then be used in order to arrive at relations which are stated purely in terms of the physical quantities. From that point on, the analysis may proceed without any reference to the RCR.

Note that the strength of the above approach is related to its limitations. We avoid divergence problems since we are not trying to cast the interacting system into the mold of a perturbation of a noninteracting "ideal gas" of connected clusters. After all, near T_c that picture fails. At the same time we lack the semblance of a full solution of the model, which is offered by expansion methods. Nevertheless, the RCR method has produced nonperturbative results which go beyond what has been accomplished by other available methods. It seems that in our favor is the fact (which is explained by the renormalization group picture) that despite the infinite number of degrees of freedom only few parameters are relevant in the critical regime. Thus much can be learned from few differential inequalities relating the key physical quantities.

3.2. The Formalism

Let us recall first the starting point for the random current representation for the symmetric ($h=0$) case. For a finite system in a region Λ, with a Hamiltonian (1.1), the partition function at $h=0$ is

$$Z = \text{trace}\left[\prod_{b \equiv \{x,y\}} e^{\beta J_b \sigma_x \sigma_y}\right] \tag{3.1}$$

where $\text{trace} = \prod_{x \in \Lambda} [\frac{1}{2} \sum_{\sigma_x = \pm 1}]$ and b stands for a pair of sites which are referred to as bonds. Expanding each exponential in powers of βJ_b, we obtain

$$Z = \sum_{\underline{n}} W(\underline{n}) \operatorname{trace}\left[\prod_{x \in \Lambda} \sigma_x^{\sum_{b \ni x} n_b}\right] \tag{3.2}$$

where $\underline{n} = (n_b)_{b \subset \Lambda}$ ranges over all the integer-valued functions on the bonds of the lattice, and

$$W(\underline{n}) = \prod_b \frac{(\beta J_b)^{n_b}}{n_b!} \tag{3.3}$$

The trace in (3.2) is zero unless all the spin functions σ_x have an even exponent, in which case it is one. Thus,

$$Z = \sum_{\partial \underline{n} = \varnothing} W(\underline{n}) \tag{3.4}$$

where the constraint is that the following set $\partial \underline{n}$, to which we refer as a set of "sources," is empty:

$$\partial \underline{n} = \left\{ x \in \Lambda \,\middle|\, \sum_{b \ni x} n_b \text{ is odd} \right\} \tag{3.5}$$

Similarly one gets the following expression for the correlation functions which are the expectation values of $\sigma_A = \prod_{x \in A} \sigma_x$, $A \subset \Lambda$,

$$\langle \sigma_A \rangle = \sum_{\partial \underline{n} = A} \frac{W(\underline{n})}{Z} \tag{3.6}$$

For ferromagnetic models, the sum is over nonnegative terms. If there is a decay of correlations this is a result of the source constraint in (3.6).

In line with the general comments made above, we shall consider the weights $W(\underline{n})$ as defining probability distributions for the collection of random variables $\{n_b\}$ within sectors characterized by source constraints. Since the source constraints express "conservation of parity," it is convenient to regard the variables $\underline{n}$ as flux numbers which describe a "current configuration."

The measures are easy to understand in the high-temperature regime where low flux numbers are favored. In such a case the typical current configurations are formed by a backbone of current lines which pairwise connect the sources in A, and some scarce eddy currents (which enhance the entropy.) As the temperature is lowered, the density of the eddy currents increases and the current lines develop larger fluctuations. The backbone is not clearly separated from the eddy currents; however, one may still regard the configuration as a combination of two such terms. In view of the exact characterization of the onset of the long-range order as a percolation transition (see Ref. 2), we expect the simple high-temperature picture to provide the right intuition for $T > T_c$. The notion of the backbone will be further developed in the next section where it leads to a random walk expansion.

The expansions (3.4), (3.6) generalize immediately to the case where a magnetic field is present. Perhaps the simplest way to incorporate it into this picture is using a construction closely related with the "ghost" spin picture of Griffiths.[19] We introduce extra "h-bonds" linking each site of the lattice with a corresponding "ghost" site (see Fig. 2). The configurations of currents can now have arbitrary sources among the h-sites (which is how we shall refer to the ghosts.) With a small abuse of notation, we shall denote by h_x both the magnetic field at x and the h-site linked to x. The corresponding h-bond is $\{x, h_x\}$. Therefore, we consider two kinds of bonds: the lattice bonds

$$B_L = \{\{x, y\} \mid x, y \in \Lambda\}$$

and the h-bonds

$$B_h = \{\{x, h_x\} \mid x \in \Lambda\}$$

More generally, if A is a set of bonds, we will denote $A_L = A \cap B_L$; $A_h = A \cap B_h$.

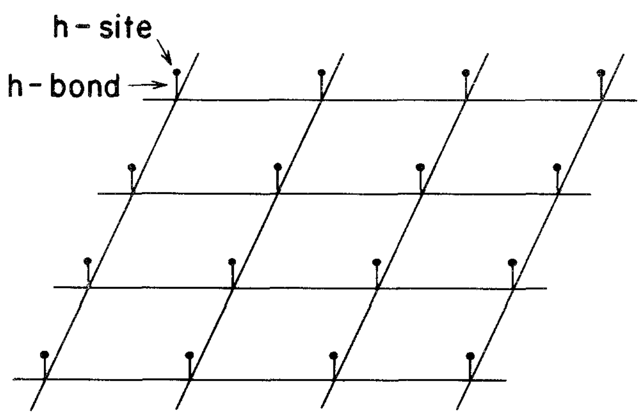


Fig. 2. Fictitious extra bonds representing the effect of a magnetic field.

The interaction with magnetic field can now be written as an Ising model with only pair interactions for which the spins at the h-sites are fixed at the value 1, and the coupling constants are

$$J_b = \begin{cases} J_{xy} & \text{for} \quad b = \{x, y\} \in B_L \\ h_x & \text{for} \quad b = \{x, h_x\} \in B_h \end{cases} \tag{3.7}$$

With this notation, we can make expressions (3.3)–(3.6) valid for the general case. In particular, to keep (3.5) we will adopt the convention that $\partial \underline{n}$ represents only the sources in Λ (lattice sources); each current configuration can have in addition an unrestricted collection of sources in h-sites (h-sources.) Again, we can interpret (3.6) probabilistically only if $J_{xy} \geqslant 0$ and $h_x \geqslant 0\ \forall x, y \in \Lambda$ (although further methods can be applied for other cases).

In order to discuss the basic techniques used to handle the RCR, we need to introduce some notation and terminology. We will say that two sites x, y (lattice- or h-sites) are *connected* by a current configuration $\underline{n}$ if there is a path of bonds with $n_b \neq 0$ joining x with y. (The definition of a path will be formalized in the next section.) In such cases we will write $\underline{n}\colon x \leftrightarrow y$. The notation $\underline{n}\colon x \to h$ will indicate that $\underline{n}$ connects x to some h-site. If A is a set of sites, $\underline{n}\colon x \to A$ means that there exists some $a \in A$ such that $\underline{n}\colon x \leftrightarrow a$. Having defined a notion of connection, we can define what a cluster is. In fact, one can define both the notion of a cluster of sites (set of connected sites), and the notion of cluster of bonds (see below.) Some confusion can arise if both kinds of clusters are used at the same time; to avoid that we shall use in this paper *only* the notion of *cluster of bonds*. Given a (lattice) site x and a current configuration $\underline{n}$, the (bond) *cluster of x in $\underline{n}$* is the set of bonds with at least one site connected to x:

$$\begin{aligned} C_{\underline{n}}(x) = \{\{y, z\} \mid y, z \in \Lambda \quad \text{and} \quad & \underline{n}\colon \\ & x \to \{y, z\}\} \cup \{\{y, h_y\} \mid y \in \Lambda \quad \text{and} \quad \underline{n}\colon x \leftrightarrow y\} \end{aligned}$$

One can also define the cluster for a set of sites; the only case of interest here is the cluster of the set of h-sites or the h-cluster:

$$\begin{aligned} C_{\underline{n}}(h) = \{\{y, z\} \mid y, z \in \Lambda \text{ and either } & \underline{n}\colon \\ & y \to h \text{ or } \underline{n}\colon z \to h\} \cup \{\{y, h_y\} \mid y \in \Lambda\} \end{aligned}$$

In addition, we need to consider the concept of connection "via" h. In a terminology motivated in the ghost spin picture, we shall say that a current configuration $\underline{n}$ *connects* two sites x, $y \in \Lambda$ *possibly via h*, and denote this by $\underline{n}\colon x \overset{h}{\leftrightarrow} y$, if either $\underline{n}\colon x \leftrightarrow y$, or $\underline{n}\colon x \to h$ and $\underline{n}\colon y \to h$.

3.3. Basic Techniques

As discussed above, one of the reasons for the power of the RCR is that it yields geometrical *identities* for some of the important physical quantities. These are obtained by means of two basic techniques: the Switching Lemma and conditioning over clusters. The Switching Lemma deals with duplicated systems of currents. An example may clarify its content. Consider two independent systems of random currents on the same lattice, one (call it blue) with $\partial \underline{n}_2 = \{x, y\}$ and another (red) with $\partial \underline{n}_1 = \{y, z\}$ (such a situation occurs in the representation of $\langle \sigma_x \sigma_y \rangle \langle \sigma_y \sigma_z \rangle$ when $h = 0$). We can picture the systems as in Fig. 3 where the broken line represents the blue currents and the unbroken line the red ones. If we add the currents, i.e., we go to a color-blind system, the picture suggests that the source y is cancelled and our original system should be closely related with a system of currents with (switched) sources $\{x, z\}$ constrained, however, to pass through y, plus another one which consists of eddy currents. The Switching Lemma states that the combinatorial factors of the RCR are such that this expected relation is an exact identity. This fact has been exploited in Ref. 16 and 2. The explicit statement of the Switching Lemma follows.

Lemma 3.1. Let $A \subset \Lambda$, $x, y \in \Lambda$ and let f be any function defined on current configurations; then

$$\sum_{\substack{\partial \underline{n}_1 = A \\ \partial \underline{n}_2 = \{x\} \Delta \{y\}}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} f(\underline{n}_1 + \underline{n}_2)$$

$$= \sum_{\substack{\partial \underline{n}_1 = A \Delta \{x\} \Delta \{y\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} f(\underline{n}_1 + \underline{n}_2)\, \mathrm{I}\, [\underline{n}_1 + \underline{n}_2 : x \overset{h}{\leftrightarrow} y] \quad (3.8)$$

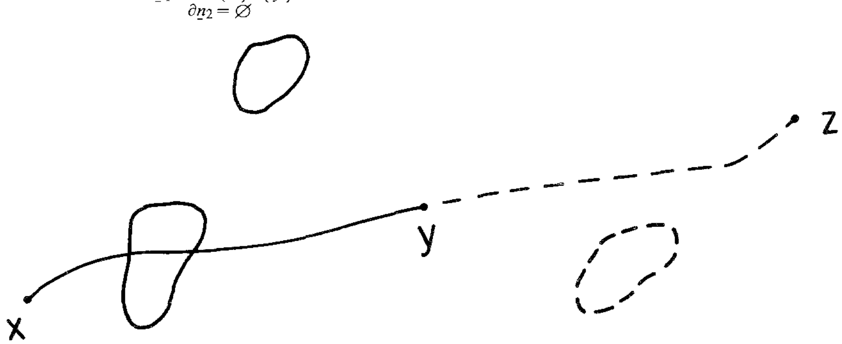


Fig. 3. An illustration for the switching principle. The two types of lines represent two independent currents. Their sum is statistically undistinguishable from the sum of two other currents whose sources are $\{x, z\}$ and $\varnothing$.

and

$$\sum_{\substack{\partial \underline{n}_1 = A \\ \partial \underline{n}_2 = \{x\}}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} f(\underline{n}_1 + \underline{n}_2)$$

$$= \sum_{\substack{\partial \underline{n}_1 = A \Delta \{x\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} f(\underline{n}_1 + \underline{n}_2) \, \mathrm{I}\,[\underline{n}_1 + \underline{n}_2 : x \to h] \tag{3.9}$$

Here $\mathrm{I}[\cdots]$ is the indicator function which takes the value 1 if the condition between the square brackets is satisfied, and 0 otherwise; Δ is the symmetric difference between sets: $A \Delta B = A \cup B \backslash (A \cap B)$; and $\underline{n}_1 + \underline{n}_2$ is defined by the bondwise sum of fluxes: $(\underline{n}_1 + \underline{n}_2)_b = n_{1b} + n_{2b}$. The lemma was proven in Ref. 2 for the case $h = 0$. The necessary adaptations of the argument when a magnetic field is present are most straightforward.

The second technique employed, conditioning over clusters, involves no combinatorics, but rather a suggestive way of rewriting certain expressions. An example of a situation to which this technique applies is depicted in Fig. 4. There is a given set of sources $A \cup B$, a site p—or a collection of sites, which could in particular be the collection of h-sites—and one is interested in summing over all current configurations subject to the constraint that p is connected to the set A and disconnected from the set B. The disconnection requirement is symbolized by the dashed line isolating p from B. This line represents $C(p)$, the (bond) cluster of p, whose (internal) boundary must be formed by bonds with zero flux num-

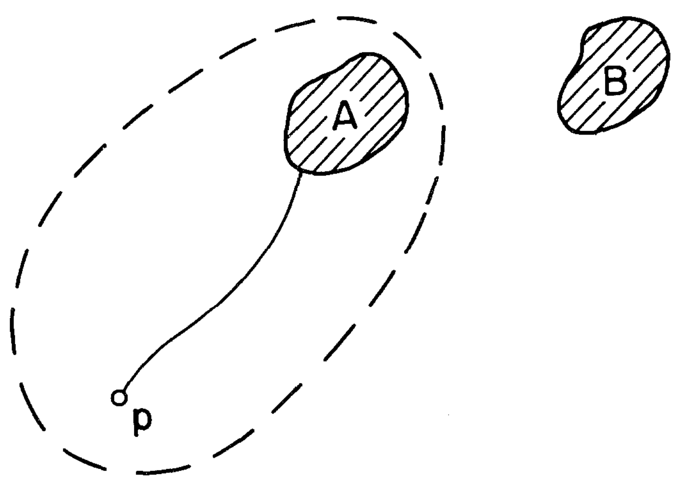


Fig. 4. The setup for the technique of conditioning over clusters.

bers. The technique consists of conditioning over the cluster $C(p)$ and, for each such cluster, summing independently over the current configurations inside and outside the cluster. The latter gives the expectation of σ_B in a less ferromagnetic system which has been deprived of all the bonds in $C(p)$. The result is:

Lemma 3.2. Consider A, $B \subset \Lambda$, $p \in \Lambda$, F a function on the subsets of B. Then:

$$\sum_{\partial \underline{n} = A \Delta B} \frac{W(\underline{n})}{Z} F(C_{\underline{n}}(p))\, \mathrm{I}\,[\underline{n}\colon p \leftrightarrow a, \forall a \in A]\, \mathrm{I}\,[\underline{n}\colon p \nleftrightarrow B]$$

$$= \sum_{\partial \underline{n} = A} \frac{W(\underline{n})}{Z} F(C_{\underline{n}}(p)\, \mathrm{I}\,[\underline{n}\colon p \leftrightarrow a, \forall a \in A] \langle \sigma_B \rangle_{C^c_{\underline{n}}(p)} \tag{3.10}$$

An analogous result holds if $C_{\underline{n}}(p)$ is replaced by $C_{\underline{n}}(h)$. We followed here the convention that if A is a set of bonds, the subindex A as in Z_A, $\langle\ \rangle_A$ indicates that the coupling constants are set to zero for bonds not belonging to A. We remark that the restriction $p \nleftrightarrow B$ is implicit in the RHS. Indeed, if there exists $a \in B$ visited by bonds of $C_{\underline{n}}(p)$ with nonzero flux, then

$$\langle \sigma_B \rangle_{C^c_{\underline{n}}(p)} = \langle \sigma_a \rangle_{\text{no interaction}} \langle \sigma_{B \setminus \{a\}} \rangle_{C^c_{\underline{n}}(p)} = 0$$

Most of the identities below involve a combination of both the Switching Lemma and conditioning over clusters. In this regard we need the analog of (3.10) for a duplicate system of currents:

Lemma 3.3. Let A_1, A_2, B_1, B_2 be subsets of Λ; $p \in \Lambda$; F function on the subsets of B. Then:

$$\sum_{\substack{\partial \underline{n}_1 = A_1 \Delta A_2 \\ \partial \underline{n}_2 = B_1 \Delta B_2}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} F(C_{\underline{n}_1 + \underline{n}_2}(p))$$
$$\times \mathrm{I}\,[\underline{n}_1 + \underline{n}_2\colon p \leftrightarrow a, \forall a \in A_1 \cup B_1]\, \mathrm{I}\,[\underline{n}_1 + \underline{n}_2\colon p \nleftrightarrow A_2 \cup B_2]$$
$$= \sum_{\substack{\partial \underline{n}_1 = A_1 \\ \partial \underline{n}_2 = B_1}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} F(C_{\underline{n}_1 + \underline{n}_2}(p))$$
$$\times \mathrm{I}\,[\underline{n}_1 + \underline{n}_2\colon p \leftrightarrow a, \forall a \in A_1 \cup B_1] \langle \sigma_{A_2} \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(p)} \langle \sigma_{B_2} \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(p)} \tag{3.11}$$

3.4. Some Applications

To demonstrate the use of the previous techniques, let us present some identities for the truncated correlation functions, which will be used later. In the following applications we need the cluster conditioning expression (3.11) for the case $p = x \in \Lambda$; $A_1 = \{x\} \Delta A$; $A_2 = B$; $(A, B \subset \Lambda)$; $F(C_{\underline{n}_1 + \underline{n}_2}(x)) = \mathrm{I}\,[\underline{n}_1 + \underline{n}_2\colon x \nleftrightarrow h]$. If we denote

$$S_x(A, B) = \sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta A \Delta B \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \leftrightarrow a \forall a \in A]$$
$$\times \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \nleftrightarrow B] \, \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \nleftrightarrow h] \tag{3.12}$$

then from (3.11),

$$S_x(A, B) = \sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta A \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \leftrightarrow a \forall a \in A]$$
$$\times \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \nleftrightarrow h] \langle \sigma_B \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(x)} \tag{3.13}$$

The first result provides an example of how the RCR takes care of the cancellations necessary for the manifestation of the sign of some of the truncated correlations.

Proposition 3.4. For any $x \in \Lambda$, $A \subset \Lambda$

$$\langle \sigma_x, \sigma_A \rangle = \sum_{\substack{A_1 \subset A \\ |A_1| \text{ odd}}} \sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta A_1 \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \leftrightarrow a \forall a \in A_1]$$
$$\times \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \nleftrightarrow h] \langle \sigma_{A \Delta A_1} \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(x)}$$
$$= \sum_{\substack{A_1 \subset A \\ |A_1| \text{ odd}}} S_x(A_1, A \Delta A_1) \tag{3.14}$$

Proof.

$$\langle \sigma_x, \sigma_A \rangle = \langle \sigma_x \sigma_A \rangle - \langle \sigma_x \rangle \langle \sigma_A \rangle$$
$$= \sum_{\partial \underline{n}_1 = \{x\} \Delta A} \frac{W(\underline{n}_1)}{Z} - \sum_{\substack{\partial \underline{n}_1 = A \\ \partial \underline{n}_2 = \{x\}}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z}$$

Now, we do the following: in the first summation of the RHS we add an extra current configuration by multiplying and dividing by $\sum_{\partial \underline{n}_2 = \varnothing} W(\underline{n}_2)$; and in the second summand we use the switching lemma (3.9) to obtain the same source distribution as in the first term. The two terms can be combined, with the result

$$\langle \sigma_x, \sigma_A \rangle = \sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta A \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \{1 - \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \to h]\}$$
$$= \sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta A \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \nleftrightarrow h]$$

For each pair $\underline{n}_1$, $\underline{n}_2$ we denote $A_1 = \{a \in A \mid \underline{n}_1 + \underline{n}_2 : x \leftrightarrow a\}$. The condition $\underline{n}_1 + \underline{n}_2 : x \nleftrightarrow h$ implies that $|A_1|$ is odd.

$$\langle \sigma_x, \sigma_A \rangle = \sum_{\substack{A_1 \subset A \\ |A_1| \text{ odd}}} \sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta A \\ \partial \underline{n}_2 = \emptyset}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \nrightarrow h]$$

$$\times \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \leftrightarrow a \forall a \in A_1] \, \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \nrightarrow A \Delta A_1]$$

$$= \sum_{\substack{A_1 \subset A \\ |A_1| \text{ odd}}} S_x(A_1, A \Delta A_1)$$

This is exactly (3.14) by Lemma 3.3. ▮

From this proposition we immediately obtain:

Corollary 3.5. For any $x, y, z \in \Lambda$

$$\langle \sigma_x, \sigma_y \rangle = \sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta \{y\} \\ \partial \underline{n}_2 = \emptyset}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \nrightarrow h] \tag{3.15}$$

$$\langle \sigma_x, \sigma_y \sigma_z \rangle = \left\{ \sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta \{y\} \\ \partial \underline{n}_2 = \emptyset}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \nrightarrow h] \langle \sigma_z \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(x)} \right\}$$

$$+ \{ y \Leftrightarrow z \text{ permutation of the above} \} \tag{3.16}$$

The results (3.15) and (3.16) can be sumarized diagrammatically as shown in Fig. 5. The diagrams are similar to that of Fig. 4 with the added convention that the interior of the cluster delimited by the broken line is

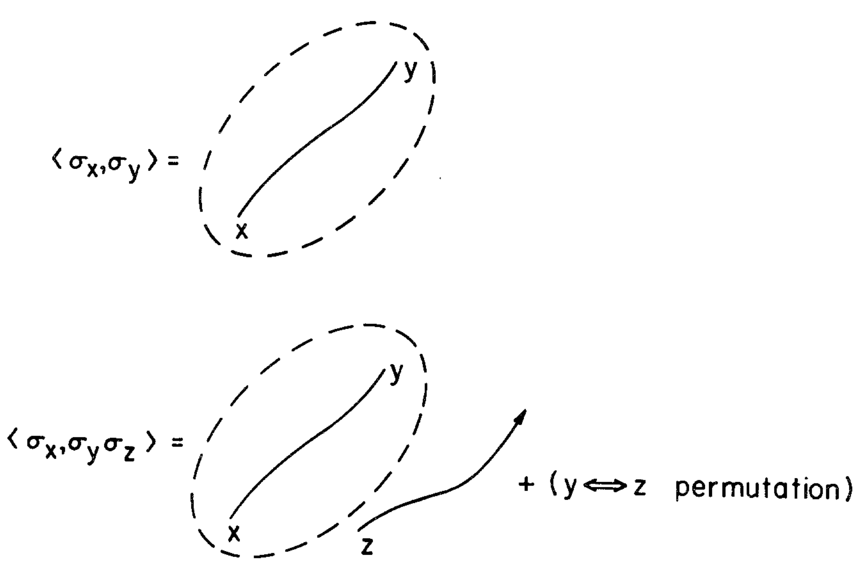


Fig. 5. Diagrammatic representations of the formulas (3.15) and (3.16).

understood to be disconnected from h-sites as well. An arrow from a given site indicates connection with h.

We now turn to analogous expressions for triple truncations.

Proposition 3.6. For any $x \in \Lambda$, A, $B \subset \Lambda$:

$$
\begin{aligned}
&\langle \sigma_x, \sigma_A, \sigma_B \rangle \\
&\quad = \Bigg\{ \sum_{\substack{A_1 \subset A \\ |A_1| \text{ odd}}} \sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta A_1 \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \leftrightarrow a \ \forall a \in A_1] \\
&\qquad \times \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \nleftrightarrow h][\langle \sigma_{A \Delta A_1} \sigma_B \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(x)} - \langle \sigma_B \rangle \langle \sigma_{A \Delta A_1} \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(x)}] \Bigg\} \\
&\qquad + \{A \Leftrightarrow B\} \\
&\qquad + \sum_{\substack{A_1 \subset A \Delta B \\ A_1 \cap A \neq \varnothing \\ A_1 \cap B \neq \varnothing \\ |A_1| \text{ odd}}} \sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta A_1 \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \leftrightarrow a \ \forall a \in A_1] \\
&\qquad \times \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \nleftrightarrow h] \langle \sigma_{A \Delta A_1} \sigma_B \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(x)}
\end{aligned} \tag{3.17}
$$

Proof.

$$
\begin{aligned}
\langle \sigma_x, \sigma_A, \sigma_B \rangle &= \langle \sigma_x, \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_x, \sigma_B \rangle - \langle \sigma_B \rangle \langle \sigma_x, \sigma_A \rangle \\
&= \sum_{\substack{A_0 \subset A \Delta B \\ |A_0| \text{ odd}}} S_x(A_0, (A \Delta B) \Delta A_0) - \langle \sigma_A \rangle \sum_{\substack{B_1 \subset B \\ |B_1| \text{ odd}}} S_x(B_1, B \Delta B_1) \\
&\quad - \langle \sigma_B \rangle \sum_{\substack{A_1 \subset A \\ |A_1| \text{ odd}}} S_x(A_1, A \Delta A_1)
\end{aligned}
$$

The mast equality is due to (3.14). To obtain (3.17) we decompose the first summand in the RHS in the form

$$
\sum_{\substack{A_0 \subset A \Delta B \\ |A_0| \text{ odd}}} = \sum_{\substack{A_0 \subset A \\ |A_0| \text{ odd}}} + \sum_{\substack{A_0 \subset B \\ |A_0| \text{ odd}}} - \sum_{\substack{A_0 \subset A \\ |A_0| \text{ odd} \\ A_0 \cap B \neq \varnothing}} - \sum_{\substack{A_0 \subset B \\ |A_0| \text{ odd} \\ A_0 \cap A \neq \varnothing}} + \sum_{\substack{A_0 \subset A \Delta B \\ |A_0| \text{ odd} \\ A_0 \cap A \neq \varnothing \\ A_0 \cap B \neq \varnothing}}
$$

and group terms, noting that in the RHS the summations preceded by a minus sign disappear from the final expression. Indeed, in all such summations the sets A, B, A_0 are such that $((A \Delta B) \Delta A_0) \cap A_0 \neq \varnothing$, hence $S_x(A_0, (A \Delta B) \Delta A_0) = 0$ by the remark following (3.10). ▌

The particular applications of this proposition which we need later are:

Corollary 3.7. For any $x, y, z, k, l \in \Lambda$

$$
\begin{aligned}
\langle \sigma_x, \sigma_y, \sigma_z \rangle
&= \Bigg\{ \sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta \{y\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}\,[\underline{n}_1 + \underline{n}_2 : x \nleftrightarrow h] \\
&\quad \times [\langle \sigma_z \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(x)} - \langle \sigma_z \rangle] \Bigg\} + \{ y \Leftrightarrow z \}
\end{aligned} \tag{3.18}
$$

$$
\begin{aligned}
\langle \sigma_x, \sigma_y \sigma_z, \sigma_k \sigma_l \rangle
&= \Bigg\{ \Bigg[\sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta \{y\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}\,[\underline{n}_1 + \underline{n}_2 : x \nleftrightarrow h] \\
&\quad \times (\langle \sigma_z \sigma_k \sigma_l \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(x)} - \langle \sigma_k \sigma_l \rangle \langle \sigma_z \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(x)}) \Bigg] + [y \Leftrightarrow z] \Bigg\} \\
&\quad + \{ (yz) \Leftrightarrow (kl) \} + \Bigg\{ \mathrm{I}\,[y, z, k, l \text{ all different}] \\
&\quad \times \Bigg[\sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta \{y, z, k\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}\,[\underline{n}_1 + \underline{n}_2 : x \nleftrightarrow h] \\
&\quad \times \mathrm{I}\,[\underline{n}_1 + \underline{n}_2 : x \leftrightarrow y.\ z \text{ and } k] \langle \sigma_l \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(x)} \Bigg] \\
&\quad + [k \Leftrightarrow l] \Bigg\} + \{ (yz) \Leftrightarrow (kl) \}
\end{aligned} \tag{3.19}
$$

4. THE RANDOM WALK REPRESENTATION

4.1. Definition of the Formalism

As discussed above, it helps to visualize the current configurations as formed by two contributions: a backbone of current lines satisfying the source constraints and a "sea" of current loops. To make this picture precise we shall define a map Ω_A which, to each current configuration $\underline{n}$ with $\partial \underline{n} = A$ (or, more generally, $\partial \underline{n} \subset A$), associates a well-defined sequence of paths ω linking the points in A between themselves or possibly with h-sites. Given such a map, we can cast (3.6) in the form

$$
\begin{aligned}
\langle \sigma_A \rangle &= \sum_{\partial \omega = A} \sum_{\partial \underline{n} = A} \frac{W(\underline{n})}{Z} \mathrm{I}\,[\Omega_A(\underline{n}) = \omega] \\
&= \sum_{\partial \omega = A} \rho(\omega)
\end{aligned} \tag{4.1}
$$

which defines a random walk representation (RWR). The key observation for the definition of Ω_A is that, due to parity constraints, the currents whose sets of sources include A must exhibit a set of disjoint walks formed by bonds where the flux is *odd*, connecting each point of A with another source. We shall describe below one possible way of constructing such a path, for which the function $\rho(\omega)$ has a number of useful properties.

The application of random walk expansions to the study of the Ising model may be traced to the work of Fisher,[20] who employed simple and self-avoiding random walks for bounds, rather than an exact representation as (4.1). The algorithm used here for the generation of a RWR is based on that of Ref. 2, in the somewhat simpler version of Ref. 11.

It may be interesting to note that the RWR presented here shares a number of properties with the random walk expansion introduced by Brydges, Fröhlich, and Spencer[12,13] for the ϕ^4 system and other continuous spin models. In a sense, the two random walk expansions are complementary: the BFS representation seems more suited for the derivation of perturbation expansions, while the one based on the Ising model deals more effectively with the "strong-coupling" limit. However, their striking similarities suggest the possibility of a unified analysis. This point is further pursued in Ref. 14. It must be emphasized, however, that pure random walk methods have not yet yielded precise enough techniques for the cancellations which are necessary for estimates of the truncated correlation functions. In this regard, the Ising model's RWR has the benefit of the additional insight provided by the underlying RCR with its powerful identities like the Switching Lemma.

Let us proceed to describe the notions which will be used in our definition of the backbone of a configuration of currents. A *step* from x to y is an oriented bond (x, y). At each site $x \in \Lambda$ we choose an order for the set of steps emerging from x, such that the step (x, h_x) is the first or earliest one. To each step (x, y) (y can be the h-site h_x), we associate a set of *cancelled* bonds formed by the bond used by the step itself and all the bonds corresponding to steps emerging from x that are earlier than (x, y). In particular, a lattice step (x, y), $y \in \Lambda$ cancels, besides itself, the step (x, h_x) and possibly other lattice steps emerging from x. On the other hand, the step (x, h_x) only cancels itself.

Definition 4.1. A sequence of steps is said to be *consistent* if no step of the sequence uses a bond cancelled by a previous step.

If ω is a consistent sequence of steps, we denote by $\tilde{\omega}$ the set of all the bonds its steps cancel. With a slight abuse of terminology we shall say that a bond b is in a sequence of steps ω (and denote this as $b \in \omega$) if one of the two steps associated with b is a step of ω. Other definitions needed are:

- A *path* or *walk* from p to q is a sequence of steps $\{(x_i, x_{i+1}), i=0,..., n\}$ with $x_0 = p$, $x_{n+1} = q$.
- Given two sequences of steps ω_1, ω_2, the *composition* $\omega_1 \circ \omega_2$ is the sequence obtained by traversing first the steps of ω_1 and then those of ω_2 (composition of sequences).
- The *lattice sources* $\partial\omega$ of a sequence of steps ω is the set of lattice points covered by an odd number of steps of ω. If $s_x(\omega)$ is the cardinality of the set $\{(p, q) \in \omega \mid p = x \text{ or } q = x\}$, then $\partial\omega = \{x \in \Lambda \mid s_x(\omega) \text{ is odd}\}$. (As for currents, the h-sources of ω will not be explicity written.)

We define the backbone of a current configuration by using the order chosen for the steps emerging from one point, and by requiring consistency—which forces the path to move forward. The first example is the following:

Definition 4.2. Given a current configuration $\underline{n}$ and a lattice site $x \in \partial\underline{n}$, the x-backbone of $\underline{n}$ is the path $\Omega_x(\underline{n}) = \{(x_i, x_{i+1}), i=0,..., n\}$ determined as follows:

(P1) $x_0 = x$ and the first step (x, x_1) is the earliest one of all the steps emerging from x with $n_{\{x,x_1\}}$ odd.

(P2) Each step (x_i, x_{i+1}) is the earliest of all steps emmerging from x_i that have not been cancelled by previous steps, and for which the flux number is odd.

(P3) The path stops when it reaches a site from which there are no more noncancelled bonds with odd flux number available. This always happen at a source of $\underline{n}$; in particular it can be at an h-site.

Note that if the backbone visits an h-site, then it must stop because the only bond available to return is the h-bond which has been cancelled by the last step. This is a basic feature of all the paths considered here: if they reach an h-site they stop there.

If more lattice sources are present we have several candidates for backbones. For instance, if there is no magnetic field the current configurations contributing to $\langle\sigma_x\sigma_y\rangle$ are (in a finite system) exactly those whose x-backbone stops at y. However, we can group current configurations in an equally effective way considering, instead of the complete x-backbone, the smaller piece obtained by cutting such backbone the first time it hits y. We found convenient to choose the second possibility, hence for a set of sources $A \subset \Lambda$ we will define the A-backbone as follows. First, we will assume a standard order for the lattice sites (for example lexicographic).

Definition 4.3. For a configuration $\underline{n}$ with $\partial\underline{n} \supset A$, the A-backbone of $\underline{n}$ is the sequence of paths $\Omega_A = \omega_1 \circ \cdots \circ \omega_s$ $(s \leqslant |A|)$ defined as follows:

(S1) Pick the earliest of the points in A, say x, and take ω_1 as the piece of the x-backbone that extends until it first hits another source of $\underline{n}$ (which can be an h-source).

(S2) To determine ω_i once $\omega_1,\dots,\omega_{i-1}$ are found, consider a system deprived of all the bonds cancelled by the sequence $\omega_1 \circ \cdots \circ \omega_{i-1}$, and remove from A all the sites already visited by such sequence. Then repeat step S1.

(S3) Continue until the set A is exhausted.

With these definitions we can write (3.6) in the form

$$\langle \sigma_A \rangle = \sum_{\partial\omega = A} \rho(\omega) \tag{4.2}$$

where, for a sequence of steps ω:

$$\rho(\omega) = \mathrm{I}\,[\omega \text{ is consistent}] \sum_{\partial\underline{n} = \partial\omega} \frac{W(\underline{n})}{Z}\, \mathrm{I}\,[\Omega_{\partial\omega}(\underline{n}) = \omega] \tag{4.3}$$

For ferromagnetic systems we have again a probabilistic picture in which the weights $\rho(\omega)/\langle\sigma_A\rangle$ define a probability distribution on the space of random sequences of steps.

We can write more explicit expressions for $\rho(\omega)$. Given a consistent sequence of steps ω, the current configurations $\underline{n}$ contributing to $\rho(\omega)$ are exactly those that satisfy:

(i) $\underline{n}$ is odd on all bonds in ω,

(ii) $\underline{n}$ is even on all bonds in $\tilde{\omega}\backslash\omega$,

(iii) $\underline{n}$ restricted to ω^c or $\tilde{\omega}^c$ is sourceless.

Therefore, using the notation (3.7), we have

$$\rho(\omega) = \mathrm{I}\,[\omega \text{ is consistent}] \prod_{b \in \omega} \sinh(\beta J_b) \prod_{b \in \tilde{\omega}\backslash\omega} \cosh(\beta J_b) \sum_{\substack{\underline{n} \text{ on } \tilde{\omega}^c \\ \partial\underline{n} = \varnothing}} \frac{W(\underline{n})}{Z}$$

or

$$\rho(\omega) = \mathrm{I}\,[\omega \text{ is consistent}] \prod_{b \in \omega} \tanh(\beta J_b) \left[\prod_{b \in \tilde{\omega}} \cosh(\beta J_b) \frac{Z_{\tilde{\omega}^c}}{Z} \right] \tag{4.4}$$

Note that the last square bracket is just the probability that all the bonds in $\tilde{\omega}$ have even flux numbers; hence:

$$\rho(\omega) = \mathrm{I}\,[\omega \text{ is consistent}] \prod_{b \in \omega} \tanh(\beta J_b) \sum_{\partial\underline{n} = \varnothing} \frac{W(\underline{n})}{Z}\, \mathrm{I}\,[n_b \text{ is even on } \tilde{\omega}] \tag{4.5}$$

4.2. Properties of the Weights

Let us list some useful properties of the weights $\rho(\omega)$.

Proposition 4.4. The weights $\rho(\omega)$ have the following properties:

(a) A simple upper bound:

$$\rho(\omega) \leqslant \mathrm{I}[\omega \text{ is consistent}] \prod_{b \in \omega} \tanh(\beta J_b) \tag{4.6}$$

(b) The composition law: If $\omega_1 \circ \omega_2$ is a consistent sequence of steps, then

$$\rho(\omega_1 \circ \omega_2) = \rho(\omega_1)\, \rho_{\bar{\omega}_1^c}(\omega_2) \tag{4.7}$$

(c) The dichotomy: For any pair of step sequences ω_1, ω_2

$\rho(\omega_1 \circ \omega_2)$ is

$$\begin{cases} \geqslant \rho(\omega_1)\, \rho(\omega_2) & \text{if} \quad \omega_1 \circ \omega_2 \text{ is consistent} \quad (4.8) \\ \equiv 0 \text{ (and hence } \leqslant \rho(\omega_1)\, \rho(\omega_2)) & \text{otherwise} \quad (4.9) \end{cases}$$

(d) If Γ is a family of sequences of steps such that there exists $A \subset \Lambda$ with $\langle \sigma_A \rangle \equiv \sum_{\omega \in \Gamma} \rho(\omega)$, then

$$\sum_{\omega_2 \in \Gamma} \rho(\omega_1 \circ \omega_2) \leqslant \rho(\omega_1) \sum_{\omega_2 \in \Gamma} \rho(\omega_2) \tag{4.10}$$

(e) Let A be a set of bonds. If $\omega \cap A = \varnothing$, then

$$\rho(\omega) \leqslant \rho_{A^c}(\omega) \tag{4.11}$$

(f) If ω does not visit h-sites

$$\rho(\omega) \leqslant \rho_{h=0}(\omega) \tag{4.12}$$

Remarks.

(i) Close analogies to the properties (a) through (d) [but not yet for (e) and (f)] are known for the Brydges–Fröhlich–Spencer random walk representation for continuous spin models. The relation (e) appears implicity in Ref. 2; however, we shall make a better use of it here.

(ii) One reason why these relations are so effective is that the direction of the inequality in the first alternative (4.8) of the dichotomy is opposite to the relation (4.10) for unconstrained sums. This fact has been used extensively in Ref. 2 and 4.

(iii) A similar complementary relation exists between the inequalities (4.10) and (4.11). Let us emphasize that the latter one requires only ω itself not to include bonds of A; $\tilde{\omega}$ need not be disjoint from A.

Proof. The above listed properties (a) up to (d) were used and discused in Ref. 2, 4, and 11. The proofs are presented here for the sake of completeness. We shall prove the properties in the order (a), (e), (f), (b), (c), (d).

Proof of (a). It follows from the observation that by (4.5) the sum in (3.4) is smaller than one.

Proof of (e). One obtains an upper bound for $\rho(\omega)$ by replacing in (4.5) the condition "n_b is even on $\tilde{\omega}$" by the weaker restriction "n_b is even on $\tilde{\omega}\backslash A$."

$$\rho(\omega) \leqslant \mathrm{I}[\omega \text{ is consistent}] \prod_{b \in \omega} \tanh(\beta J_b) \sum_{\partial \underline{n} = \varnothing} \frac{W(\underline{n})}{Z} \mathrm{I}[n_b \text{ is even on } \tilde{\omega}\backslash A]$$

$$= \mathrm{I}[\omega \text{ is consistent}] \prod_{b \in \omega} \tanh(\beta J_b) \prod_{b \in \tilde{\omega}\backslash A} \cosh(\beta J_b) \frac{Z_{(\tilde{\omega}\backslash A)^c}}{Z} \tag{4.13}$$

On the other hand, from (4.4):

$$\rho_{A^c}(\omega) = \mathrm{I}[\omega \text{ is consistent}] \prod_{b \in \omega} \tanh(\beta J_b) \prod_{b \in \tilde{\omega}\backslash A} \cosh(\beta J_b) \frac{Z_{(\tilde{\omega}^c\backslash A)}}{Z_{A^c}}$$

Hence

$$\rho(\omega) \leqslant \rho_{A^c}(\omega) \frac{Z_{A^c}}{Z_{(\tilde{\omega}^c\backslash A)}} \frac{Z_{(\tilde{\omega}\backslash A)^c}}{Z}$$

$$= \rho_{A^c}(\omega) \frac{Z_{A^c}}{Z_{(A^c\backslash \tilde{\omega})}} \frac{Z_{A \cup (A^c\backslash \tilde{\omega})}}{Z} \tag{4.14}$$

(4.11) is therefore a consequence of the following inequality:

Claim. If A, A' are disjoints families of bonds, then, for a ferromagnetic system,

$$\frac{Z_{A^c}}{Z} \leqslant \frac{Z_{A'}}{Z_{A \cup A'}} \tag{4.15}$$

The LHS can be interpreted as the change in the partition function produced by turning off the bonds in A, while the RHS is the change when one turns off the bonds of A in a system after the bonds of $(A \cup A')^c$ have already been turned off. From this point of view (4.15) means that there is a smaller effect in the latter case. To prove (4.15), notice that

$$\text{LHS} = \left[\left\langle \exp\left(\sum_{b \in A} J_b \sigma_b\right)\right\rangle_{A^c}\right]^{-1} \leqslant \left[\left\langle \exp\left(\sum_{b \in A} J_b \sigma_b\right)\right\rangle_{A'}\right]^{-1} = \text{RHS}$$

by the Griffiths (II) inequality.

Proof of (f). Set $A = B_h$ in (4.11).

Proof of (b). Let us first remark that if $\omega_1 \circ \omega_2$ is a consistent path, then $\tilde{\omega}_1 \cap \omega_2 = \varnothing$. This is so because ω_2 is chosen out of a system for which all the bonds in $\tilde{\omega}_1$ were removed (Property S2 in Definition 4.3). Therefore, every current configuration $\underline{n}$ contributing to $\rho(\omega_1 \circ \omega_2)$ can be uniquely written as $\underline{n} = \underline{n}_1 + \underline{n}_2$ with $\underline{n}_1$ supported on $\tilde{\omega}_1$ and $\underline{n}_2$ on $\tilde{\omega}_1^c$. One has that $\partial \underline{n}_1 = \partial \omega_1$, $\partial \underline{n}_2 = \partial \omega_2$, and, since $\underline{n}_1$ and $\underline{n}_2$ have disjoint supports, $W(\underline{n}) = W(\underline{n}_1)\, W(\underline{n}_2)$. We obtain (4.7) by conditioning on $\tilde{\omega}_1$.

Proof of (c). (4.8) follows from (4.7) and (4.11). (4.9) is trivial because the LHS is zero.

Proof of (d). In the LHS only the terms with $\omega_1 \circ \omega_2$ consistent contribute. For these we can use (4.7)

$$\begin{aligned}
\sum_{\omega_2 \in \Gamma} \rho(\omega_1 \circ \omega_2) &= \rho(\omega_1) \sum_{\omega_2 \in \Gamma} \rho_{\tilde{\omega}_1^c}(\omega_2)\, \mathrm{I}[\omega_1 \circ \omega_2 \text{ consistent}] \\
&\leqslant \rho(\omega_1) \sum_{\omega_2 \in \Gamma} \rho_{\tilde{\omega}_1^c}(\omega_2) = \rho(\omega_1) \langle \sigma_A \rangle_{\tilde{\omega}_1^c} \\
&\leqslant \rho(\omega_1) \langle \sigma_A \rangle = \rho(\omega_1) \sum_{\omega_2 \in A} \rho(\omega_2)
\end{aligned}$$

where we used the Griffiths (II) inequality. ∎

4.3. The kernel *K*

Within the random walk formalism it is quite natural to introduce a certain "kernel" which although lacking direct physical meaning is revealing for the analysis of the critical behavior. A kernel of this type was first introduced in the work of Fröhlich and Sokal[7] within the framework of the BFS representation. It was shown there to be a useful tool for the derivation of an inequality which, for instance, leads to the bound (1.7). In recognition of the fact that the kernel introduced below shares some properties with the one of Fröhlich and Sokal[7] we shall use for it the same notation K.

Definition 4.5. For $x, y \in \Lambda$:

$$K(x, y) = \sum_{\partial \underline{n} = \{x, y\}} \frac{W(\underline{n})}{Z} \mathrm{I}\left[\Omega_x(\underline{n}) \text{ visits } y\right] \tag{4.16}$$

In words, $K(x, y)$ is formed by those current configurations with sources in $\{x, y\}$ whose x-backbone is not stopped at an h-site before visiting y, that is, whose $\{x, y\}$-backbone actually connects x with y. For such configurations the step sequence $\Omega_{xy}(\underline{n})$ of Definition 4.3 consists of a "path" from x to y [the other alternative in (4.1) is that Ω_A is a pair of paths from x and y to h]. Therefore,

$$K(x, y) = \sum_{\omega : x \to y} \rho(\omega) \tag{4.17}$$

where we adopted the notation

$$\omega : x \to y \tag{4.18}$$

to mean that ω is a (not necessarily consistent) path which starts at x, ends at y, and visits y only *once* (i.e., it stops when it reaches y). In (4.18) y can be either a lattice or an h-site. Moreover, the expression

$$\omega : x \to h \tag{4.19}$$

will mean that ω starts at x and ends at some h-site. For instance,

$$\langle x \rangle = \sum_{\omega : x \to h} \rho(\omega) = \sum_{y} \sum_{\omega : x \to h_y} \rho(\omega) \tag{4.20}$$

The reason for the relevance of K is that it shows up in the probabilities for walks to pass through a given site or to use a certain bond. An example of this role for K is provided by the next proposition.

Proposition 4.6. For any sites $x, y \in \Lambda$ and z which is either a lattice site or an h-site, let $\Gamma_{(x,y,z)} = \{$paths ω which start at x and whose last step is $(y, z)\}$. Then

$$\sum_{\omega \in \Gamma_{(x,y,z)}} \rho(\omega) \leqslant K(x, y) \tanh(\beta J_{yz}) \tag{4.21}$$

Proof. Splitting from the paths their last step we have

$$\begin{aligned}
&\sum_{\omega \in \Gamma_{(x,y,z)}} \rho(\omega) \\
&\quad = \sum_{\partial \underline{n} = \{x,z\}} \frac{W(\underline{n})}{Z} \mathrm{I}\left[\Omega_x(\underline{n}) \text{ reaches } z \text{ for the last time via the step } (y, z)\right] \\
&\quad \leqslant \sum_{\partial \underline{n} = \{x,z\}} \frac{W(\underline{n})}{Z} \mathrm{I}\left[n_{y,z} \text{ is odd}\right] \mathrm{I}\left[\Omega_x(\underline{n}) \text{ visits } y\right]
\end{aligned}$$

We can now sum independently over odd values of $n_{y,z'}$ and over configurations $\underline{n}'$ defined on $B\backslash\{y, z\}$ with sources $\partial\underline{n}' = \{x, y\}$ and satisfying the restriction of the last indicator function. The sum over $n_{y,z}$ yields a hyperbolic sine which can be replaced by a hyperbolic tangent by switching to a sum over *even* values of $n_{y,z}$. In conclusion,

$$\sum_{\omega \in \Gamma(x,y,z)} \rho(\omega) \leqslant \tanh(\beta J_{yz}) \sum_{\partial\underline{n} = \{x,y\}} \frac{W(\underline{n})}{Z} \mathrm{I}[n_{y,z} \text{ is even}] \times \mathrm{I}[\Omega_x(\underline{n}) \text{ visits } y]$$

Relaxing the restriction on the parity of $n_{y,z}$, we obtain (4.21). ∎

Since the kernel K is a natural object within the random walk formalism, it is important to find expressions for it in terms of physical quantities. The next two propositions provide some basic bounds. Formally, these relations are closely analogous to those which first appeared in the work of Fröhlich and Sokal[7] within the BFS formalism. One difference is that here (4.24) is not quite an equality. Another is the existence of the very useful upper bound in (4.22).

Proposition 4.7.

$$\langle \sigma_x ; \sigma_y \rangle \leqslant K(x, y) \leqslant \langle \sigma_x \sigma_y \rangle_{h=0} \tag{4.22}$$

Proof.

$$\begin{aligned} \langle \sigma_x ; \sigma_y \rangle &= \langle \sigma_x \sigma_y \rangle - \langle \sigma_x \rangle \langle \sigma_y \rangle \\ &= \sum_{\omega: x \to y} \rho(\omega) + \sum_{\substack{\omega_1 : x \to h \\ \omega_2 : y \to h}} [\rho(\omega_1 \circ \omega_2) - \rho(\omega_1)\,\rho(\omega_2)] \\ &= K(x, y) + \Delta(x, y) \end{aligned} \tag{4.23}$$

As $\sum_{\omega_2 : x \to h} \rho(\omega_2) = \langle y \rangle$, we can apply (4.10) to prove that $\Delta(x, y) \leqslant 0$, which proves the leftmost inequality in (4.22). The rightmost inequality is an immediate consequence of (4.17), and (4.12). ∎

Let us note that, except for a narrow wedge around the domain R_0 in the $(\beta, \beta h)$ plane, there would be a very significant difference between the upper and lower bound in (4.22). The following result provides a much sharper physical representation for the sum of the kernel K.

Proposition 4.8.

$$\sum_{y \in \Lambda} K(x, y) \frac{\tanh(\beta h_y)}{1 + \langle \sigma_y \rangle \tanh(\beta h_y)} \leqslant \langle \sigma_x \rangle \leqslant \sum_{y \in \Lambda} K(x, y) \tanh(\beta h_y) \tag{4.24}$$

Note that the two bounds differ only by a factor of $1 + O(\beta h)(\simeq 1)$.

Proof. The upper bound is just (4.21) when z is replaced by the h-site h_z. To prove the lower bound we start from

$$\langle\sigma_x\rangle = \sum_y \sum_{\omega:\, x\to h_y} \rho(\omega)$$

Each ω: $x \to h_y$ is of the form $\omega = \omega_L \circ (y, h_y)$ with $\omega_L: x \to y$, i.e., ω_L only visits y once. Moreover, $\tilde\omega = \tilde\omega_L \cup \{y, h_y\}$. Therefore

$$\begin{aligned}
\langle\sigma_x\rangle &= \sum_y \tanh(\beta h_y) \sum_{\omega_L:\, x\to y} \prod_{b\in\omega_L} \tanh(\beta J_b) \prod_{b\in\tilde\omega_L} \cosh(\beta J_b) \\
&\quad \times \cosh(\beta h_y) \frac{Z_{(\tilde\omega_L \cup \{y,h_y\})^c}}{Z} \\
&= \sum_y \tanh(\beta h_y) \sum_{\omega_L:\, x\to y} \rho(\omega_L) \frac{\cosh(\beta h_y)\, Z_{\tilde\omega_L^c \setminus \{y,h_y\}}}{Z_{\tilde\omega_L^c}} \\
&= \sum_y \tanh(\beta h_y) \sum_{\omega_L:\, x\to y} \rho(\omega_L) \frac{\cosh(\beta h_y)}{\langle \exp(\beta h_y \sigma_y)\rangle_{\tilde\omega_L^c \setminus \{y,h_y\}}}
\end{aligned} \tag{4.25}$$

But

$$\begin{aligned}
\langle \exp(\beta h_y \sigma_y)\rangle_{A^c} &= \cosh(\beta h_y) + \langle\sigma_y\rangle_{A^c} \sinh(\beta h_y) \\
&\leqslant \cosh(\beta h_y) + \langle\sigma_y\rangle \sinh(\beta h_y)
\end{aligned}$$

We have used Griffiths (II) inequality. Substituting this into (4.25) we obtain (4.24). ∎

For the applications below we will suppose translation invariance. In this case (4.24) yields

$$\frac{M}{\tanh(\beta h)} \leqslant \sum_y K(x, y) \leqslant \frac{M}{\tanh(\beta h)} [1 + M \tanh(\beta h)] \tag{4.26}$$

5. PROOF OF THE DIFFERENTIAL INEQUALITIES

We turn now to the actual proof of the main differential inequalities (1.15) and (1.16). In this section we assume that the system is translation invariant and has a finite $|J| = \sum_x J_{0x} < \infty$.

5.1. Some Preliminary Results

To free the proofs of details which could obscure the flow of the argument, we have grouped in this section certain auxiliary results. Most of these results are bounds of physical quantities in terms of objects defined in the random current or random walk ensembles. The only exception is Lemma 5.2 which provides a bound of certain truncation functions in terms of simpler correlations.

In the sequel we will make use of a particular application of Lemma 3.3, which is stated next. It is obtained by setting $F=1$, $A_1=\varnothing$, $B_1=\varnothing$ in (3.11).

Lemma 5.1. For A, B, $\subset \Lambda$; $p \in \Lambda$:

$$\sum_{\substack{\partial \underline{n}_1 = A \\ \partial \underline{n}_2 = B}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : p \nleftrightarrow A \cup B] = E\{\langle \sigma_A \rangle_{C^c_{\underline{n}_1+\underline{n}_2}(p)} \langle \sigma_B \rangle_{C^c_{\underline{n}_1+\underline{n}_2}(p)}\} \tag{5.1}$$

where E is the probability measure on the space of duplicated sourceless current configurations defined by

$$E\{f(\underline{n}_1+\underline{n}_2)\} = \sum_{\substack{\partial \underline{n}_1 = \varnothing \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} f(\underline{n}_1+\underline{n}_2) \tag{5.2}$$

An analogous result holds if $C^c_{\underline{n}_1+\underline{n}_2}(p)$ is replaced by $C^c_{\underline{n}_1+\underline{n}_2}(h)$.

Note that for $A=\{x\} \Delta \{y\}$, $B=\varnothing$, and p replaced by h, (5.1) yields another expression for the two-point truncated correlation:

$$\langle \sigma_x, \sigma_y \rangle = E\{\langle \sigma_x \sigma_y \rangle_{C^c_{\underline{n}_1+\underline{n}_2}(h)}\} \tag{5.3}$$

The next result is a new correlation inequality which is derived using the random current representation.

Proposition 5.2. For any $x, y, z, k, l \in \Lambda$

$$\begin{aligned}\langle \sigma_x, \sigma_y\sigma_z, \sigma_k\sigma_l \rangle \leqslant \{&[2\langle \sigma_x, \sigma_y \rangle \langle \sigma_k \sigma_z \rangle_{h=0} \langle \sigma_l \rangle + (k \Leftrightarrow l)] \\ &+ [(yz) \Leftrightarrow (kl)]\} + \tfrac{1}{2}\{y \Leftrightarrow z\}\end{aligned} \tag{5.4}$$

For coincidental pairs $\{k, l\} = \{y, z\}$ we also have

$$\langle \sigma_x, \sigma_y\sigma_z, \sigma_y\sigma_z \rangle \leqslant 0 \tag{5.5}$$

Proof. (5.5) follows from the Griffiths (II) inequality:

$$\langle \sigma_x, \sigma_y\sigma_z, \sigma_y\sigma_z \rangle = -2\langle \sigma_y\sigma_z \rangle \langle \sigma_x, \sigma_y\sigma_z \rangle \leqslant 0$$

To prove (5.4) we resort to (3.19) which we write as

$$\langle \sigma_x, \sigma_y\sigma_z, \sigma_k\sigma_l \rangle = \Delta_1 + \Delta_2 + \text{Permut.} \tag{5.6}$$

where Δ_1 and Δ_2 are the two summatories in the RHS of (3.19).

To bound Δ_1 we need to obtain a bound, for any set of bonds A, of

$$\begin{aligned}
&\langle \sigma_z \sigma_k \sigma_l \rangle_A - \langle \sigma_k \sigma_l \rangle \langle \sigma_z \rangle_A \\
&\quad = \langle \sigma_z, \sigma_k \sigma_l \rangle_A - [\langle \sigma_k \sigma_l \rangle - \langle \sigma_k \sigma_l \rangle_A] \langle \sigma_z \rangle_A \\
&\quad \leqslant \langle \sigma_z, \sigma_k \sigma_l \rangle_A
\end{aligned} \tag{5.7}$$

The neglected term is nonpositive as a consequence of the Griffiths (II) inequality. Resorting once more to this inequality we have, from (3.16),

$$\begin{aligned}
&\langle \sigma_z, \sigma_k \sigma_l \rangle \\
&\quad \leqslant \langle \sigma_1 \rangle \sum_{\substack{\partial \underline{n}_1 = \{z\} \Delta \{k\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \nleftrightarrow h] + (l \Leftrightarrow k) \\
&\quad = \langle \sigma_1 \rangle \langle \sigma_z, \sigma_k \rangle + (l \Leftrightarrow k) \\
&\quad \leqslant \langle \sigma_l \rangle \langle \sigma_z \sigma_k \rangle_{h=0} + (l \Leftrightarrow k)
\end{aligned} \tag{5.8}$$

We have used (3.15) in the equality and (4.22) in the last inequality. This bound is monotonic with the strength of the interaction, hence

$$\langle \sigma_z, \sigma_k \sigma_l \rangle_A \leqslant \langle \sigma_l \rangle \langle \sigma_z \sigma_k \rangle_{h=0} + (l \Leftrightarrow k)$$

Combining this with (5.7) for $A = C^c_{\underline{n}_1 + \underline{n}_2}(x)$ we finally get

$$\begin{aligned}
\Delta_1 &\leqslant \langle \sigma_l \rangle \langle \sigma_z \sigma_k \rangle_{h=0} \sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta \{y\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \nleftrightarrow h] \\
&\quad + (l \Leftrightarrow k) \\
&= \langle \sigma_l \rangle \langle \sigma_z \sigma_k \rangle_{h=0} \langle \sigma_x, \sigma_y \rangle + (l \Leftrightarrow k)
\end{aligned} \tag{5.9}$$

The last step is due to (3.15).

To bound Δ_2 we ignore the requirement that y, z, k, l be all different, use the Switching Lemma (3.9) and Griffiths (II) inequality, and replace the restriction "$\underline{n}_1 + \underline{n}_2 : x \to z$" by the weaker one "$\underline{n}_1 + \underline{n}_2 : z \nleftrightarrow h$." We obtain

$$\begin{aligned}
\Delta_2 &\leqslant \langle \sigma_l \rangle \sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta \{y\} \\ \partial \underline{n}_2 = \{z\} \Delta \{k\}}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \nleftrightarrow h] \, \mathrm{I}[\underline{n}_1 + \underline{n}_2 : z \nleftrightarrow h] \\
&= \langle \sigma_l \rangle \, E\{ \langle \sigma_x \sigma_y \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(h)} \langle \sigma_z \sigma_k \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(h)} \}
\end{aligned}$$

The equality is due to (5.1). We pull out one correlation outside the E-integration by means of the bound $\langle \sigma_z \sigma_k \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(h)} \leqslant \langle \sigma_z \sigma_k \rangle_{h=0}$. The remaining E-integration is just $\langle \sigma_x, \sigma_y \rangle$ by (5.3). In conclusion,

$$\Delta_2 \leqslant \langle \sigma_l \rangle \langle \sigma_z \sigma_k \rangle_{h=0} \langle \sigma_x, \sigma_y \rangle \tag{5.10}$$

This is exactly the same contribution as (5.9), except that the permutation $(l \Leftrightarrow k)$ is missing.

We obtain (5.5) by substituting (5.9) and (5.10) in (5.6) and suitably grouping the permutations. ∎

Lemma 5.3.

$$\sum_{y,z \in \Lambda} \sum_{\substack{\partial \underline{n}_1 = \{0\} \Delta \{y\} \\ \partial \underline{n}_2 = \{y\} \Delta \{z\}}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : y \nrightarrow h] \geqslant \chi^2 \tag{5.11}$$

Remark. The result (5.11) is a direct consequence of the following inequalite due to Graham[21]

$$\sum_{\substack{\partial \underline{n}_1 = \{0\} \Delta \{y\} \\ \partial \underline{n}_2 = \{y\} \Delta \{z\}}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : y \nrightarrow h] \geqslant \langle \sigma_0, \sigma_y \rangle \langle \sigma_y, \sigma_z \rangle$$

However, the case of interest here, in which the sites y and z are summed over, admits a simpler proof which we present for the sake of completeness. In the proof we use the fact that if Λ is a finite group (under the addition $+$) and $f: \Lambda^3 \to R$ a function such that $f(x+a, y+a, z+a) = f(x, y, z)$ for every $x, y, z, a \in \Lambda$, then

$$\sum_{y,z \in \Lambda} f(x, y, z) = \sum_{y,z \in \Lambda} f(y, x, z) \tag{5.12}$$

for every $x \in \Lambda$. Indeed, both expressions are equal to $(1/|\Lambda|) \sum_{x,y,z} f(x, y, z)$.

Proof. Let us first use (5.12) and then (5.1)

$$\begin{aligned} \mathrm{LHS} &= \sum_{y,z \in \Lambda} \sum_{\substack{\partial \underline{n}_1 = \{0\} \Delta \{y\} \\ \partial \underline{n}_2 = \{0\} \Delta \{z\}}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : 0 \nrightarrow h] \\ &= E\left\{ \sum_{y,z \in \Lambda} \langle \sigma_0 \sigma_y \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(h)} \langle \sigma_0 \sigma_z \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(h)} \right\} \\ &= E\left\{ \left[\sum_{y \in \Lambda} \langle \sigma_0 \sigma_y \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(h)} \right]^2 \right\} \end{aligned}$$

We now apply the Schwartz inequality in $L^2(dE)$ which yields

$$\mathrm{LHS} \geqslant \left[E\left\{ \sum_{y \in \Lambda} \langle \sigma_0 \sigma_y \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(h)} \right\} \right]^2 = \left[\sum_{y \in \Lambda} \langle \sigma_0, \sigma_y \rangle \right]^2 = \chi^2$$

In the first equality we used (5.3). ∎

Lemma 5.4. For $x, y, u, l \in \Lambda$ and $h=0$

$$\begin{aligned}
&\langle\sigma_x\sigma_l\rangle\langle\sigma_l\sigma_u\rangle\langle\sigma_u\sigma_y\rangle \\
&\qquad\leqslant \sum_{\substack{\partial \underline{n}_1=\{x\}\Delta\{u\}\\ \partial\underline{n}_2=\{u\}\Delta\{y\}}} \frac{W(\underline{n}_1)}{Z}\frac{W(\underline{n}_2)}{Z}\,\mathrm{I}[\underline{n}_1+\underline{n}_2: x\to l] \\
&\qquad\leqslant \langle\sigma_x\sigma_l\rangle\langle\sigma_l\sigma_u\rangle\langle\sigma_u\sigma_y\rangle + (x\Leftrightarrow y)
\end{aligned}\tag{5.13}$$

Remark. This result provides an example of how, in the framework of the RCR, the intuitive picture which applies in high temperatures may suggest relations which remain basically true at all temperatures. The sum in (5.13) is over pairs of configurations $\underline{n}_1, \underline{n}_2$ with the sources $\{x, u\}$ and $\{u, y\}$ correspondingly. At high temperature one expects the required connections to be performed in the minimal way, in which case each of these configurations would consist of a simple path (ω_1 and ω_2 respectively) linking the sources, and only rare fluctuations. The condition $\mathrm{I}[\underline{n}_1+\underline{n}_2: x\to l]$ would be satisfied if either the path ω_1 or the path ω_2 visit the site l. The contribution from these two possibilities perturbatively looks like the upper bound in (5.13). Our result shows that the above relation holds as an inequality even on a nonperturbative level.

Let us remark also that the lower bound in (5.13) shows that the upper bound there is off by at most a factor of 2.

Proof.

$$\begin{aligned}
\mathrm{LHS} &= \sum_{\substack{\partial \underline{n}_1=\{x\}\Delta\{u\}\\ \partial\underline{n}_2=\{u\}\Delta\{y\}}} \frac{W(\underline{n}_1)}{Z}\frac{W(\underline{n}_2)}{Z}\{1-\mathrm{I}[\underline{n}_1+\underline{n}_2: x\not\leftrightarrow l]\} \\
&= \langle\sigma_x\sigma_u\rangle\langle\sigma_u\sigma_y\rangle - E\{\langle\sigma_x\sigma_u\rangle_{C^c_{\underline{n}_1+\underline{n}_2}(l)}\,\langle\sigma_u\sigma_y\rangle_{C^c_{\underline{n}_1+\underline{n}_2}(l)}\}
\end{aligned}$$

The last expression is due to (5.1). Adding and substracting $\langle\sigma_u\sigma_y\rangle\, E\{\langle\sigma_x\sigma_u\rangle_{C^c_{\underline{n}_1+\underline{n}_2}(l)}\}$ we obtain

$$\begin{aligned}
\mathrm{LHS} &= \langle\sigma_u\sigma_y\rangle\, E\{\langle\sigma_x\sigma_u\rangle - \langle\sigma_x\sigma_u\rangle_{C^c_{\underline{n}_1+\underline{n}_2}(l)}\} \\
&\quad + E\{[\langle\sigma_u\sigma_y\rangle - \langle\sigma_u\sigma_y\rangle_{C^c_{\underline{n}_1+\underline{n}_2}(l)}]\langle\sigma_x\sigma_u\rangle_{C^c_{\underline{n}_1+\underline{n}_2}(l)}\}
\end{aligned}\tag{5.14}$$

To obtain the upper bound we bound the last term in the RHS by a term analogous to the first one by using a Griffiths inequality. Then, we use (5.1) in both terms to obtain

$$\text{LHS} \leqslant \langle \sigma_u \sigma_y \rangle \sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta \{u\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \{1 - \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \nleftrightarrow l]\} + (x \Leftrightarrow y)$$

$$= \langle \sigma_u \sigma_y \rangle \sum_{\substack{\partial \underline{n}_1 = \{x\} \Delta \{u\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : x \leftrightarrow l] + (x \Leftrightarrow y)$$

$$= \langle \sigma_u \sigma_y \rangle \sum_{\substack{\partial \underline{n}_1 = \{l\} \Delta \{u\} \\ \partial \underline{n}_2 = \{x\} \Delta \{l\}}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} + (x \Leftrightarrow y)$$

$$= \langle \sigma_u \sigma_y \rangle \langle \sigma_l \sigma_u \rangle \langle \sigma_x \sigma_l \rangle + (x \Leftrightarrow y)$$

Along the way we used the Switching Lemma (3.8).

To obtain the lower bound we simply neglect the last summand in the RHS of (5.14) and proceed as above. ∎

The last preliminary inequality can be entirely derived from the properties proved for the random walk formalism.

Lemma 5.5. Let A be a set of bonds; $A = A_L \cup A_h$; and let $\Pi(A_h)$ denote the set of the lattice sites of the bonds of A_h: $\Pi(A_h) = \{k \in \Lambda: \{k, h_k\} \in A_h\}$. Then, for every $v \in \Lambda$,

$$\langle \sigma_v \rangle - \langle \sigma_v \rangle_{A^c} \leqslant \sum_{k \in \Pi(A_h)} \tanh(\beta h_k)\, K(v, k)$$

$$+ \sum_{(k,l):\, \{k,l\} \in A_L} K(v, k) \tanh(\beta J_{kl}) \langle \sigma_l \rangle_{A^c} \tag{5.15}$$

Proof. Using the random walk representation (4.20):

$$\langle \sigma_v \rangle - \langle \sigma_v \rangle_{A^c} = \sum_{\omega:\, v \to h} \rho(\omega) - \sum_{\substack{\omega:\, v \to h \\ \omega \cap A = \varnothing}} \rho_{A^c}(\omega)$$

$$\leqslant \sum_{\omega:\, v \to h} \rho(\omega) - \sum_{\substack{\omega:\, v \to h \\ \omega \cap A = \varnothing}} \rho(\omega)$$

$$= \sum_{\substack{\omega:\, v \to h \\ \omega \cap A \neq \varnothing}} \rho(\omega)$$

The inequality is due to (4.11). The paths ω intercepting A can be classified in terms of the last step they take in A. If such step involves a bond in A_h the path is of the form $\omega: v \to h_k$ for some $k \in \Pi(A_h)$; if the last step in A is

(k, l) with $\{k, l\} \in A_{L'}$ the path is of the form $\omega = \omega_1 \circ \omega_2$ with ω_1 having (k, l) as last step; and $\omega_2 : l \to h$ with $\omega_2 \cap A = \varnothing$. Hence

$$\begin{aligned}\langle \sigma_v \rangle - \langle \sigma_v \rangle_{A^c} &\leqslant \sum_{k \in \Pi(A_h)} \sum_{\omega :\, v \to h_k} \rho(\omega) + \sum_{(k,l):\, \{k,l\} \in A_L} \sum_{\substack{\omega_1 \in \Gamma(v,k,l) \\ \omega_2 : l \to h \\ \omega_2 \cap A = \varnothing}} \rho(\omega_1 \circ \omega_2) \\ &= \mathrm{I} + \mathrm{II} \end{aligned} \tag{5.16}$$

The situation is more easily understood in the pictorial form of Fig. 6. The proof consists in splitting the last step (k, h_k) from the walks contributing to the first diagram (term I in (5.16)); and the intermediate step (k, l) from the walks of the second diagram (term II).

Indeed, the inequality (4.21) implies

$$\mathrm{I} \leqslant \sum_{k \in \Pi(A_L)} \tanh(\beta h_k)\, K(v, k) \tag{5.17}$$

which is the first summand in the RHS of (5.15). In II we will first use (4.7) and then the representation (4.20):

$$\begin{aligned}\mathrm{II} &= \sum_{(k,l):\, \{k,l\} \in A_L} \sum_{\omega_1 \in \Gamma(v,k,l)} \rho(\omega_1) \sum_{\substack{\omega_2 : l \to h \\ \omega_2 \cap A = \varnothing}} \rho_{\bar{\omega}_1^c}(\omega_2) \\ &\leqslant \sum_{(k,l):\, \{k,l\} \in A_L} \sum_{\omega_1 \in \Gamma(v,k,l)} \rho(\omega_1) \langle \sigma_l \rangle_{A^c \cap \bar{\omega}_1^c}\end{aligned}$$

We use the bound $\langle \sigma_l \rangle_{A^c \cap \bar{\omega}_1^c} \leqslant \langle \sigma_l \rangle_{A^c}$ and we resort once again to (4.21)

$$\mathrm{II} \leqslant \sum_{(k,l):\, \{k,l\} \in A_L} K(v, k) \tanh(\beta J_{kl}) \langle \sigma_l \rangle_{A^c} \tag{5.18}$$

This is the last summand in the RHS of (5.15). ∎

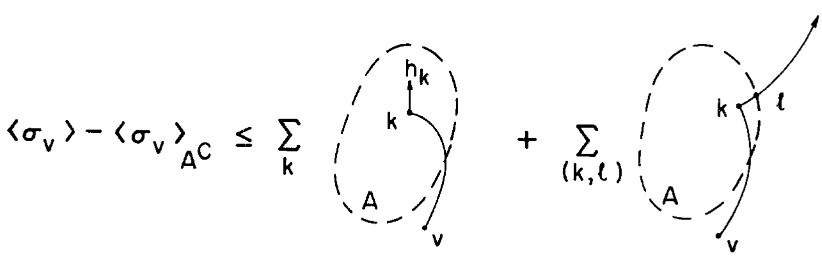


Fig. 6. Diagrammatic representation of the walks which contribute in the upper bound (5.16).

5.2. Differential Inequalities

Theorem 5.6. In a finite, translation-invariant (i.e., periodic) ferromagnetic Ising system, we have

$$\frac{\partial M}{\partial \beta} \geqslant \frac{|M\,|J|\,\chi - \tanh(\beta h)\,|J|\,B_0 \chi|_+}{1+2\beta\,|J|\,B_0} \tag{5.19}$$

Proof.

$$\frac{\partial M}{\partial \beta} = \frac{1}{2} \sum_{u,v \in \Lambda} J_{uv} \langle \sigma_0, \sigma_u \sigma_v \rangle \tag{5.20}$$

Therefore, from (3.16),

$$\frac{\partial M}{\partial \beta} = \frac{1}{2} \sum_{u,v \in \Lambda} J_{uv} \left\{ \left[\sum_{\substack{\partial \underline{n}_1 = \{0\} \Delta \{u\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \times \mathrm{I}[\underline{n}_1 + \underline{n}_2 : 0 \nrightarrow h] \langle \sigma_v \rangle_{C^c_{\underline{n}_1+\underline{n}_2}(0)} \right] + [u \Leftrightarrow v] \right\} \tag{5.21}$$

By symmetry the sum for the permuted term is identical to the sum for the first term, hence its net effect is to remove the factor 1/2.

The basic idea of the proof can be explained with the aid of diagrams. Starting with the diagram in Fig. 7, we can write $\partial M/\partial \beta$ as the difference of the terms with no restriction for the walk emerging from v and the ones in which this walk is constrained to intercept the cluster of 0. These are exactly the diagrams on Fig. 6 with two extra sources inside the cluster limited by the dashed line (which really represents a hypersurface).

Such diagrams suggest the inequality

$$\frac{\partial M}{\partial \beta} \geqslant (M - \tanh(\beta h)\, B_0)\, |J|\, \frac{\partial M}{\partial(\beta h)} - 2\beta B_0\, |J|\, \frac{\partial M}{\partial \beta} \tag{5.22}$$

which is indeed proven below. Let us note that had not been for the manifest presence of the factor $\partial M/\partial \beta$ in the RHS which allows us to close

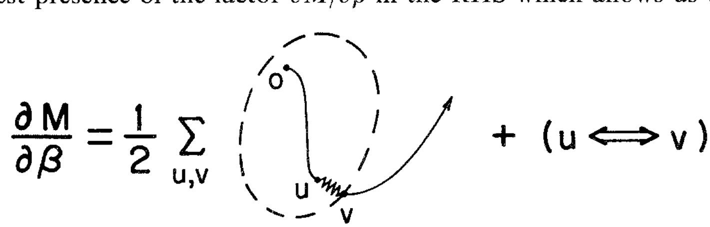


Fig. 7. A diagrammatic representation of the current configurations which contribute in (5.21). The solid lines indicate the backbones, the dashed line denotes the existence of a separating surface of zero flux, and the wiggled short line represents the numerical factor J_{uv}.

the inequality, the relation (5.22) would have been of little use since the RHS can easily be negative.

Let us now derive (5.22) by a detailed analysis. We start by bounding $\langle\sigma_v\rangle_{C^c_{\underline{n}_1+\underline{n}_2}(0)}$ by the expression obtained from (5.15):

$$\begin{aligned}\frac{\partial M}{\partial\beta} \geqslant \sum_{u,v\in\Lambda} J_{uv} \sum_{\substack{\partial\underline{n}_1=\{0\}\Delta\{u\}\\ \partial\underline{n}_2=\varnothing}} \frac{W(\underline{n}_1)}{Z}\frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1+\underline{n}_2: 0 \nrightarrow h]\\ \times\left\{\langle\sigma_v\rangle - \tanh(\beta h)\sum_{k\in\Pi(C_{\underline{n}_1+\underline{n}_2}(0))} K(v,k)\right.\\ \left. - \sum_{(k,l):\,\{k,l\}\in C_{\underline{n}_1+\underline{n}_2}(0)} K(v,k)\tanh(\beta J_{kl})\langle\sigma_l\rangle_{C^c_{\underline{n}_1+\underline{n}_2}(0)}\right\}\\ = \mathrm{I}-\mathrm{II}-\mathrm{III}\end{aligned} \tag{5.23}$$

By translation invariance we obtain

$$\mathrm{I} = \sum_u \langle\sigma_0, \sigma_u\rangle \sum_v J_{uv}\langle\sigma_v\rangle = M\,|J|\,\chi \tag{5.24}$$

We rearrange the second term in the form

$$\begin{aligned}\mathrm{II} = \tanh(\beta h)\sum_{u,v,k} K(v,k)\,J_{uv} \sum_{\substack{\partial\underline{n}_1=\{0\}\Delta\{u\}\\ \partial\underline{n}_2=\varnothing}} \frac{W(\underline{n}_1)}{Z}\frac{W(\underline{n}_2)}{Z}\\ \times \mathrm{I}[\underline{n}_1+\underline{n}_2: 0\nrightarrow h]\,\mathrm{I}[\underline{n}_1+\underline{n}_2: 0\rightarrow k]\\ = \tanh(\beta h)\sum_{u,v,k} K(v,k)\,J_{uv} E\{\langle\sigma_u\sigma_k\rangle_{C^c_{\underline{n}_1+\underline{n}_2}(h)}\langle\sigma_0\sigma_k\rangle_{C^c_{\underline{n}_1+\underline{n}_2}}\}\end{aligned} \tag{5.25}$$

In the last step we used the switching lemma (3.8) and (5.1). By Griffiths (II) inequality $\langle\sigma_u\sigma_k\rangle_{C^c_{\underline{n}_1+\underline{n}_2}(h)} \leqslant \langle\sigma_u\sigma_k\rangle_{h=0}$. Hence

$$\begin{aligned}\mathrm{II} \leqslant \tanh(\beta h)\sum_{u,v,k} K(v,k)\,J_{uv}\langle\sigma_u\sigma_k\rangle_{h=0}\,E\{\langle\sigma_0\sigma_k\rangle_{C^c_{\underline{n}_1+\underline{n}_2}(h)}\}\\ = \tanh(\beta h)\sum_k \langle\sigma_0, \sigma_k\rangle\left[\sum_{u,v} J_{uv} K(v,k)\langle\sigma_u\sigma_k\rangle_{h=0}\right]\end{aligned}$$

The equality is due to (5.3). To bound the sum inside the square bracket we first use the rightmost inequality in (4.22) and then the Schwartz inequality in the space of functions $f(u, v)$ which are square summable with the weights J_{uv}:

$$\begin{aligned}
&\sum_{u,v} J_{uv} K(v,k) \langle \sigma_u \sigma_k \rangle_{h=0} \\
&\quad \leqslant \left\{ \sum_{u,v} J_{uv} [\langle \sigma_v \sigma_k \rangle_{h=0}]^2 \right\}^{1/2} \left\{ \sum_{u,v} J_{uv} [\langle \sigma_u \sigma_k \rangle_{h=0}]^2 \right\}^{1/2} \\
&\quad = B_0 \, |J|
\end{aligned} \tag{5.26}$$

Hence,

$$\mathrm{II} \leqslant \tanh(\beta h) \, \chi B_0 \, |J| \tag{5.27}$$

We shall now operate on the third term in (5.23) in order to pull a factor $\partial M / \partial \beta$ which can be grouped with the LHS. We first notice that the restriction $\{k, l\} \in C_{\underline{n}_1 + \underline{n}_2}(0)$ is, in the presence of the factor $\langle \sigma_1 \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(0)}$, equivalent to the condition "$\underline{n}_1 + \underline{n}_2 : 0 \to k$." Hence,

$$\begin{aligned}
\mathrm{III} \leqslant \sum_{u,v,k,l} K(v,k) \, J_{uv} \beta J_{kl} \Bigg\{ &\sum_{\substack{\partial \underline{n}_1 = \{0\} \Delta \{u\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \\
&\times \mathrm{I}[\underline{n}_1 + \underline{n}_2 : 0 \nrightarrow h] \, \mathrm{I}[\underline{n}_1 + \underline{n}_2 : 0 \leftrightarrow k] \langle \sigma_1 \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(0)} \Bigg\}
\end{aligned} \tag{5.28}$$

(where we also used the bound $\tanh x \leqslant x$). The sum in the curly brackets can be written in terms of the h-cluster via the following equality:

Claim. For any $a, b, c, d \in \Lambda$

$$\begin{aligned}
&\sum_{\substack{\partial \underline{n}_1 = \{a\} \Delta \{b\}, \\ \partial \underline{n}_2 = \{a\} \Delta \{c\}}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \, \mathrm{I}[\underline{n}_1 + \underline{n}_2 : a \nrightarrow h] \langle \sigma_d \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(a)} \\
&\quad = \sum_{\substack{\partial \underline{n}_1 = \{d\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \langle \sigma_a \sigma_b \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(h)} \langle \sigma_a \sigma_c \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(h)}
\end{aligned} \tag{5.29}$$

The claim is a consequence of Lemma 3.3 which implies that each of the expressions in (5.29) is equal to

$$\sum_{\substack{\partial \underline{n}_1 = \{a\} \Delta \{b\} \Delta \{d\} \\ \partial \underline{n}_2 = \{a\} \Delta \{c\}}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \, \mathrm{I}[\underline{n}_1 + \underline{n}_2 : \{a, b, c\} \nrightarrow h] \, \mathrm{I}[\underline{n}_1 + \underline{n}_2 : d \to h]$$

After applying the Switching Lemma (3.8) and (5.29) the curly bracket in (5.28) becomes

$$\{\ \} = \sum_{\substack{\partial \underline{n}_1 = \{l\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \langle \sigma_u \sigma_k \rangle_{C^c_{\underline{n}_1+\underline{n}_2}(h)} \langle \sigma_0 \sigma_k \rangle_{C^c_{\underline{n}_1+\underline{n}_2}(h)}$$

$$\leqslant \langle \sigma_u \sigma_k \rangle_{h=0} \sum_{\substack{\partial \underline{n}_1 = \{l\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \langle \sigma_0 \sigma_k \rangle_{C^c_{\underline{n}_1+\underline{n}_2}(h)}$$

where the inequality is a consequence of Griffiths (II). Now we use again (5.29) (for $a = c$):

$$\{\ \} \leqslant \langle \sigma_u \sigma_k \rangle_{C^c_{\underline{n}_1+\underline{n}_2}(h)} \sum_{\substack{\partial \underline{n}_1 = \{0\} \Delta \{k\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : 0 \nleftrightarrow h] \langle \sigma_l \rangle_{C^c_{\underline{n}_1+\underline{n}_2}(h)}$$

$$= \langle \sigma_u \sigma_k \rangle_{h=0} \langle \sigma_0, \sigma_k \sigma_l \rangle$$

[see (3.16)]. Substituting this into (5.28), we obtain

$$\mathrm{III} \leqslant \sum_{k,l} \left[\sum_{u,v} J_{uv} K(v, k) \langle \sigma_u \sigma_k \rangle_{h=0} \right] \beta J_{kl} \langle \sigma_0, \sigma_k \sigma_l \rangle$$

For the square bracket we use the bound (5.26) and the rest is 2β times the first summand in (5.20):

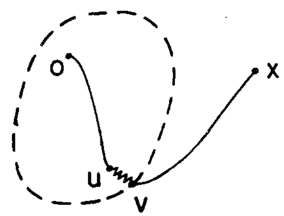
$$\mathrm{III} \leqslant B_0 \, |J| \, 2\beta \frac{\partial M}{\partial \beta} \tag{5.30}$$

Substituting (5.24), (5.27), and (5.30) in (5.23), we obtain (5.22). To prove the theorem we just have to solve for $\partial M / \partial \beta$. ▮

Remark. Using the same arguments, but starting from the diagram of Fig. 8, one obtains in the $h = 0$ case:

$$\frac{\partial \chi}{\partial \beta} \geqslant \frac{|J| \, \chi^2}{1 + 2\beta \, |J| \, B_0} \tag{5.31}$$

$$\frac{\partial \chi}{\partial \beta} = \frac{1}{2} \sum_{x,u,v}$$



$$+ (u \Longleftrightarrow v)$$

Fig. 8. Diagrammatic representation of the quantity bounded by inequality (5.31).

This inequality was first obtained by Aizenman and Graham[3] in a less direct way (which, however, yielded also other results).

Theorem 5.7. In a finite, translation-invariant ferromagnetic Ising system $\partial M/\partial(\beta h)$ satisfies the following bounds:

(a)

$$\left|\frac{\partial \chi}{\partial(\beta h)}\right| \leqslant 4\beta\, |J|\, [1+\tfrac{1}{2}Q][1+M\tanh(\beta h)]^3 \frac{M^4}{[\tanh(\beta h)]^3} \tag{5.32}$$

(b) For any $\beta \geqslant \beta_c/24$

$$\left|\frac{\partial \chi}{\partial(\beta h)}\right| \geqslant \frac{|1-Q\bar{B}_0|_+^2}{48[1+M\tanh(\beta h)]\, B_0(1+2\beta\, |J|\, B_0)^2} \tanh(\beta h)\, \chi^4 \tag{5.33}$$

As in (1.15), $Q(h, t) = \tanh(\beta h)/(M\, |J|\, \beta)$ and $\bar{B}_0 = B_0\, |J|\, \beta$.

Remark. The upper bound is mentioned here only for completeness, the main result is the lower bound which differs from the upper one by the appropriate bubble insertions.

Proof. Our basic expression is a consequence of (3.18):

$$\begin{aligned}\left|\frac{\partial \chi}{\partial(\beta h)}\right| &= \sum_{x,y\in\Lambda} |\langle \sigma_0, \sigma_x, \sigma_y\rangle| \\ &= 2\sum_{x,y} \sum_{\substack{\partial \underline{n}_1 = \{0\}\Delta\{x\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}\,[\underline{n}_1 + \underline{n}_2 : 0 \nleftrightarrow h] \\ &\quad \times (\langle \sigma_y\rangle - \langle \sigma_y\rangle_{C^c_{\underline{n}_1+\underline{n}_2}(0)})\end{aligned} \tag{5.34}$$

(a) Proof of the upper bound

Inserting (5.15) in the RHS of (5.34), we obtain

$$\begin{aligned}\left|\frac{\partial \chi}{\partial(\beta h)}\right| &\leqslant 2\sum_{x,y} \sum_{\substack{\partial \underline{n}_1 = \{0\}\Delta\{x\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}\,[\underline{n}_1 + \underline{n}_2 : 0 \nleftrightarrow h] \\ &\quad \times \Big\{\sum_k \tanh(\beta h)\, K(y,k)\, \mathrm{I}\,[\underline{n}_1 + \underline{n}_2 : 0 \leftrightarrow k] + \beta M \sum_{l,k} K(y,k)\, J_{kl} \\ &\quad \times (\mathrm{I}\,[\underline{n}_1 + \underline{n}_2 : 0 \leftrightarrow k] + \mathrm{I}\,[\underline{n}_1 + \underline{n}_2 : 0 \leftrightarrow l])\Big\}\end{aligned} \tag{5.35}$$

where the following bounds were used:

(i) $\langle \sigma_y \rangle_{C^c_{\underline{n}_1+\underline{n}_2}(0)} \leqslant M$

(ii) $\tanh x \leqslant x$

(iii) $\mathrm{I}[\{k, l\} \in C_{\underline{n}_1+\underline{n}_2}(0)] \leqslant \mathrm{I}[\underline{n}_1+\underline{n}_2 : 0 \leftrightarrow k] + \mathrm{I}[\underline{n}_1+\underline{n}_2 : 0 \leftrightarrow l]$

The random current sums in (5.35) may be simplified by the following bound, in terms of the kernel K.

$$\sum_{\substack{\partial \underline{n}_1 = \{0\} \Delta \{x\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}\,[\underline{n}_1+\underline{n}_2 : 0 \nleftrightarrow h]\, \mathrm{I}\,[\underline{n}_1+\underline{n}_2 : 0 \leftrightarrow k]$$
$$= \sum_{\substack{\partial \underline{n}_1 = \{k\} \Delta \{x\} \\ \partial \underline{n}_2 = \{0\} \Delta \{k\}}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}\,[\underline{n}_1+\underline{n}_2 : 0 \nleftrightarrow h] \leqslant K(x, k)\, K(0, k) \tag{5.36}$$

The equality is due to the Switching Lemma and the inequality is because all the terms in the LHS satisfy the condition that both $\Omega_x(\underline{n}_1)$ and $\Omega_0(\underline{n}_2)$ visit k. With this, (5.35) becomes

$$\left| \frac{\partial \chi}{\partial(\beta h)} \right| \leqslant 2 \sum_{\substack{x,y\\k}} \Big\{ [\tanh(\beta h) + \beta M\, |J|]\, K(y, k)\, K(0, k)\, K(x, k)$$
$$+ \beta M \sum_l J_{kl} K(y, k)\, K(0, l)\, K(x, l) \Big\}$$

To arrive at (5.32) we now need only to use (4.26).

(b) Proof of the lower bound

The basic strategy of the proof was suggested by the technique used by Brydges, Fröhlich, and Sokal[13] for the derivation of skeleton inequalities, and it consists of repeated application of the fundamental theorem of calculus. The systems studied here differ, however, from the "softly" coupled fields analyzed there. For the latter, each iteration produces a term of higher order in the bare coupling constant λ_0, hence the succesive terms can be controlled if the coupling is sufficiently weak. For the "infinitely coupled" Ising model, the expansion parameter is instead the bubble B_0 which may be extremely large in the region we are interested in. Thus, we do not obtain satisfactory bounds by just truncating the procedure at a finite number of steps. In a sense, we must consider all orders of B_0 at the same time. One of the ways to do that was showed in the previous theorem, where the terms were handled so that a bothersome factor $(1-B_0)$ was transformed into $(1+B_0)^{-1}$. For the present theorem we resort to still another technique which is the dilution method used in a different context in Ref. 22. It may be of interest to see a systematic elaboration of these methods used here in a somewhat *ad hoc* fashion.

The starting point for the lower bound on $|\partial\chi/\partial(\beta h)|$ is the equality (5.34) where the last factor in the RHS represents the effect on $\langle\sigma_y\rangle$ of turning off the pair couplings along the bonds of the cluster $C_{\underline{n}_1+\underline{n}_2}(0)$. Let us now "dilute" this cluster by replacing it by its intersection with a set of bonds A which would be described later.

By the Griffiths (II) monotonicity, for each set of bonds A:

$$\langle\sigma_y\rangle - \langle\sigma_y\rangle_{C^c_{\underline{n}_1+\underline{n}_2}(0)} \geqslant \langle\sigma_y\rangle - \langle\sigma_y\rangle_{(C_{\underline{n}_1+\underline{n}_2}(0)\cap A)^c} \tag{5.37}$$

Let us now invoke the fundamental theorem of calculus. Our first application is

$$\langle\sigma_y\rangle - \langle\sigma_y\rangle_{(C_{\underline{n}_1+\underline{n}_2}(0)\cap A)^c} = \int_0^1 ds \frac{\partial}{\partial s}\langle\sigma_y\rangle_s \tag{5.38}$$

where $\langle\cdots\rangle_s$ means average in a system for which the coupling constant for each $\{u,v\}\in C_{\underline{n}_1+\underline{n}_2}(0)\cap A$ has been multiplied by s. Therefore

$$\begin{aligned}&\langle\sigma_y\rangle - \langle\sigma_y\rangle_{(C_{\underline{n}_1+\underline{n}_2}(0)\cap A)^c}\\ &\quad\geqslant \frac{1}{2}\sum_{u,v}\beta J_{uv}\,\mathrm{I}[\underline{n}_1+\underline{n}_2: 0\to u]\,\mathrm{I}[\{u,v\}\in A]\int_0^1 ds\langle\sigma_y, \sigma_u\sigma_v\rangle_s\end{aligned} \tag{5.39}$$

We have used the fact that $\mathrm{I}[\{u,v\}\in C_{\underline{n}_1+\underline{n}_2}(0)] \geqslant \mathrm{I}[\underline{n}_1+\underline{n}_2: 0\leftrightarrow u]$.

We would like to use now the lower bound provided by the previous theorem for the truncated correlation function which appears in (5.39). However, this function was modified by the parameter s. To control its effect we apply the fundamental theorem of calculus once again:

$$\begin{aligned}\langle\sigma_y, \sigma_u\sigma_v\rangle_s &= \langle\sigma_y, \sigma_u\sigma_v\rangle - \int_s^1 ds_1 \frac{\partial}{\partial s_1}\langle\sigma_y, \sigma_u\sigma_v\rangle_{s_1}\\ &= \langle\sigma_y, \sigma_u\sigma_v\rangle - \frac{1}{2}\sum_{k,l}\beta J_{kl}\,\mathrm{I}[\{k,l\}\in C_{\underline{n}_1+\underline{n}_2}(0)]\,\mathrm{I}[\{k,l\}\in A]\\ &\quad\times\int_s^1 ds_1\langle\sigma_y, \sigma_u\sigma_v, \sigma_h\sigma_l\rangle_{s_1}\end{aligned} \tag{5.40}$$

No further iteration of this tactic is needed; we have bounds in the right direction for the truncated correlations appearing in the above expression, and we are able to control the remaining factors appearing in (5.34). Substituting the decomposition (5.40) into (5.39) and then using the result in (5.34), we find that for every set of bonds A:

$$\left|\frac{\partial\chi}{\partial(\beta h)}\right| \geqslant S_1(A) - S_2(A) \tag{5.41}$$

with

$$S_1(A) = \frac{1}{2} \sum_{x,y,u,v} \beta J_{uv} \langle \sigma_y, \sigma_u \sigma_v \rangle \, \mathrm{I}[\{u, v\} \in A] \, T(0, u, x) \tag{5.42}$$

$$S_2(A) = \frac{1}{4} \sum_{\substack{x,y,u,v\\k,l}} \beta J_{uv} \beta J_{kl} \, \mathrm{I}[\{u, v\} \in A] \, \mathrm{I}[\{k, l\} \in A] \, R(0, x, y, u, v, k, l)$$
$$\times \int_0^1 ds_1 \int_{s_1}^1 ds_2 \langle \sigma_y, \sigma_u \sigma_v, \sigma_k \sigma_l \rangle_{s_2} \tag{5.43}$$

where

$$\begin{aligned} &T(0, u, x) \\ &\quad = \sum_{\substack{\partial \underline{n}_1 = \{0\} \Delta \{x\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : 0 \nleftrightarrow h] \, \mathrm{I}[\underline{n}_1 + \underline{n}_2 : 0 \leftrightarrow u] \\ &R(0, x, y, u, v, k, l) \\ &\quad = \sum_{\substack{\partial \underline{n}_1 = \{0\} \Delta \{x\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathrm{I}[\underline{n}_1 + \underline{n}_2 : 0 \nleftrightarrow h] \, \mathrm{I}[\underline{n}_1 + \underline{n}_2 : 0 \to u] \end{aligned} \tag{5.44}$$

$$\times \mathrm{I}[\{k, l\} \in C_{\underline{n}_1 + \underline{n}_2}(0)] \tag{5.45}$$

The inequality (5.41) is satisfied for every set of bonds A. We shall now average it over random sets generated by selecting bonds independently with probability p, whose choice will be specified later. The corresponding averages of the indicator function which appear in (5.42) are

$$\begin{aligned} P(\mathrm{I}[\{u, v\} \in A]) &= p \\ P(\mathrm{I}[\{u, v\} \in A] \, \mathrm{I}[\{k, l\} \in A]) &= p^2 \qquad \text{if} \quad \{u, v\} \neq \{k, l\} \end{aligned} \tag{5.46}$$

What is accomplished is that the term S_1 acquires a factor p, while S_2 picks the much smaller factor of p^2, except for its coincidental terms which pick a factor p. Since, by (5.5), such terms are negative, we still obtain an upper bound on S_2 by ignoring the last distinction. With a suitable choice of p, the difference $P(S_1) - P(S_2)$ would be positive, and large enough for our purpose.

Let us now consider $P(S_1)$. After averaging over A, the sum over y and v of the relevant factors in (5.42) yields [by (5.12)] the following simple value—independently of u and x.

$$\frac{1}{2} \sum_{y,v} J_{uv} \langle \sigma_y, \sigma_u \sigma_v \rangle = \frac{\partial M}{\partial \beta}$$

Hence, using the bound on $\partial M/\partial \beta$ provided by Theorem 5.6

$$P(S_1) \geqslant \frac{|M\, |J|\, \chi - \tanh(\beta h)\, |J|\, B_0 \chi|_+}{1 + 2\beta\, |J|\, B_0}\, p\beta \sum_{u,x} T(0, u, x) = pC_1 \quad (5.47)$$

The sum over $T(0, u, x)$ also has a rather simple lower bound:

$$\sum_{u,x} T(0, u, x) \geqslant \chi^2 \quad (5.48)$$

which is proven in Lemma 5.3. However, we shall use it only after the terms $P(S_1)$ and $-P(S_2)$ are combined together.

Let us consider now $P(S_2)$. To deal with R we bound

$$\mathrm{I}[\{k, l\} \in C_{\underline{n}_1 + \underline{n}_2}(0)] \leqslant \mathrm{I}[\underline{n}_1 + \underline{n}_2 : 0 \leftrightarrow k] + \mathrm{I}[\underline{n}_1 + \underline{n}_2 : 0 \leftrightarrow l]$$

and decompose R into a sum over clusters $C^c_{\underline{n}_1 + \underline{n}_2}(h)$. Using the switching lemma (3.8) outside each cluster, one gets

$$R = E\left\{ \sum_{\substack{\partial \underline{n}_1^1 = \{0\} \Delta \{u\} \\ \partial \underline{n}_2^1 = \{u\} \Delta \{x\}}} \frac{W(\underline{n}_1^1)}{Z} \frac{W(\underline{n}_2^1)}{Z}\, \mathrm{I}[\underline{n}_1^1 + \underline{n}_2^1 : 0 \leftrightarrow k] \right\} + (k \Leftrightarrow l)$$

where $\underline{n}_1^1$, $\underline{n}_2^1$ are the current configurations on $C^c_{\underline{n}_1 + \underline{n}_2}(h)$. In order to relate R to the factor T which appears in S_1, we apply the bound (5.13) together with the inequality $\langle \sigma_0 \sigma_k \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(h)} \leqslant \langle \sigma_0 \sigma_k \rangle_{h=0}$ to obtain

$$R \leqslant [\langle \sigma_0 \sigma_k \rangle_{h=0}\, T(0, u, x) + (u \Leftrightarrow k)] + [k \Leftrightarrow l] \quad (5.49)$$

Substituting in (5.43) the factors p^2, from the average (5.46), and the relation (5.49) we have

$$P(S_2) \leqslant p^2 \sum_{\substack{x, y, u, v \\ k, l}} \beta J_{uv} \beta J_{kl} \langle \sigma_0 \sigma_k \rangle_{h=0}\, T(0, u, x)$$

$$\times \int_0^1 ds_1 \int_{s_1}^1 ds_2 \langle \sigma_y, \sigma_u \sigma_v, \sigma_k \sigma_l \rangle_{s_2} \quad (5.50)$$

[The permutations in (5.49) were used to cancel the factor 1/4 in (5.43).]

The third-order truncated correlation function in (5.50) is bounded by our new correlation inequality, which was presented in Proposition 5.2. Altogether, the bound (5.4) involves 12 terms, which can be controlled by only slight variations of the same argument. As an example, let us present here one of them, which is

$$U = p^2 \sum_{\substack{x,y,u,v\\k,l}} T(0, u, x)\, \beta J_{uv} \beta J_{kl} \langle \sigma_0 \sigma_k \rangle_{h=0}$$

$$\times \int_0^1 ds_1 \int_{s_1}^1 ds_2 \langle \sigma_l \rangle_{s_2} \langle \sigma_y, \sigma_u \rangle_{s_2} \langle \sigma_v \sigma_k \rangle_{s_2, h=0}$$

The two untruncated correlation functions in the integral are bounded by its values with $s_2 = 1$ [Griffiths (II)]. An upper bound for the truncated correlation $\langle \sigma_y, \sigma_u \rangle$ that is monotone in s_2 is obtained by combining the leftmost inequality of (4.22) with the rightmost inequality of (4.26). With these bounds,

$$U = p^2 \beta^2 \frac{M^2}{\tanh(\beta h)} (1 + M \tanh(\beta h)) \sum_{x,u} T(0, u, x) \sum_v J_{uv}$$

$$\times \sum_{k,l} J_{kl} \langle \sigma_0 \sigma_k \rangle_{h=0} \langle \sigma_v \sigma_k \rangle_{h=0}$$

$$\leqslant p^2 \beta^2 \frac{M^2}{\tanh(\beta h)} (1 + M \tanh(\beta h)) |J|^2 B_0 \sum_{x,u} T(0, u, x)$$

$$= p^2 C_2 \tag{5.51}$$

In the last inequality we have used the Schwartz inequality as in (5.26). In fact, each of the 12 terms in the above-mentioned bound for $P(S_2)$ is also bounded by $p^2 C_2$. Thus $P(S_2) \leqslant 12 p^2 C_2$, and hence, by substituting this inequality and (5.47) into (5.41), we have

$$\left| \frac{\partial \chi}{\partial h} \right| \geqslant p C_1 - 12 p^2 C_2$$

The major difference between the magnitudes C_1 and C_2 [defined in (5.47) and (5.51)] is that C_2 has also the factor B_0. Therefore, C_2 may become much larger than C_1 at the critical regime. It is to cure this problem that we resorted to the dilution technique in the first place. Indeed, if we optimize this inequality by choosing the maximizing value of p, namely $p = c_1/24 c_2$, we obtain

$$\left| \frac{\partial \chi}{\partial (\beta h)} \right| \geqslant \frac{\left| 1 - \dfrac{\tanh(\beta h)\, B}{M} \right|_+^2}{48 [1 + M \tanh(\beta h)]\, B_0 (1 + 2\beta\, |J|\, B_0)^2} \tanh(\beta h)\, \chi^2 \sum_{u,x} T(0, u, x)$$

We note that since $B_0 \geqslant 1$ and β_c is well known to obey the mean field bound $\beta_c |J| \geqslant 1$, the condition $\beta \geqslant \beta_c/24$ is sufficient to ensure that such p is not larger than one.

Having performed the cancellation between $P(S_1)$ and $P(S_2)$, we finally apply the bound (5.48) on $\sum_{u,x} T(0, u, x)$. The resulting inequality is the lower bound (5.33). ∎

This concludes the derivation of the differential inequalities. The results summarized in Sec. 1 are based on the lower bound (5.33) of the last theorem, as discussed in Sec. 2.

APPENDIX A. BOUNDS ON THE CRITICAL BEHAVIOR OF THE BUBBLE

We summarize here a set of sufficient conditions which lead to the bounds (1.8) on the bubble. In this section we assume translation invariance and denote by $G(p)$ the Fourier transform of $\langle \sigma_x, \sigma_y \rangle$.

The derivation of (1.8) is based on the following four properties of the model, which refer only to the regime $R_0 = \{\beta < \beta_c, h = 0\}$:

1. "Gaussian domination" bounds:

$$G(p) \leqslant \frac{1}{2\beta E(p)} \qquad \text{for} \quad \beta < \beta_c \tag{A.1}$$

with

$$E(p) = \frac{1}{2} \sum_x (1 - e^{ipx}) \, J_{0x} \tag{A.2}$$

which is the energy of an elementary excitation in a "spin wave" picture. Such bounds are known to be satisfied in reflection positive models.[9]

2. The bound

$$|G(p)| \leqslant \chi \tag{A.3}$$

which follows from the positivity of $\langle \sigma_x, \sigma_y \rangle$ [implied, for instance, by the Griffiths (II) inequality].

3. A differential inequality of the form

$$-\frac{\partial \chi^{-1}}{\partial \beta} \geqslant \frac{|J|}{c_1 + c_2(\beta\, |J|)^a \, B_0} \tag{A.4}$$

with $a \geqslant 1$. Such an inequality is satisfied for Ising models with $c_1 = 1$, $c_2 = 2$, $a = 1$; and for general models in the Griffiths–Simon class with $c_1 = 3/2$, $c_2 = 2$, $a = 2$ [Ref. 3; see also (5.31)].

4. An upper bound on the integrated density of states of low "spin wave excitations." We refer by this to the quantity $\kappa(u)$ defined as

$$\kappa(u) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} dp\, \mathrm{I}\,[E(p) \leqslant u] \tag{A.5}$$

Let us note that u is naturally restricted to $0 \leqslant u \leqslant |J|$ by (A.2). The assumption which is needed is that

$$\kappa(u) \leqslant c(J) \left(\frac{u}{|J|}\right)^s \qquad \text{with some } s > 1 \tag{A.6}$$

For the nearest neighbor Ising, and ϕ^4, models

$$E(p) \geqslant c(J)\, p^2 \Rightarrow \kappa(u) \leqslant \bar{c}(J)\, u^{d/2} \tag{A.7}$$

and hence the assumption (A.6) is satisfied in $d > 2$ dimensions. For long-range one-dimensional models with $J_{0x} = C\,|x|^{-\lambda}$, we have

$$E(p) \geqslant c(C)\, p^{\lambda - 1} \Rightarrow \kappa(u) \leqslant \bar{c}(C)\, u^{1/(\lambda - 1)} \tag{A.8}$$

and thus (A.6) is satisfied if $\lambda < 2$.

Lemma A.1. In a system satisfying the above four assumptions, i.e., (A.1), (A.3), (A.4), and (A.6), the magnetic susceptibility and the bubble diagram satisfy the following bounds along the line $h = 0$, $\beta \leqslant \beta_c$:

$$\chi \leqslant \begin{cases} C & s > 2 \\ Ct^{-1}[1 + |\ln t|] & s = 2 \\ Ct^{-1/(s-1)} & 1 < s < 2 \end{cases} \tag{A.9}$$

$$B_0 \leqslant \begin{cases} C & s > 2 \\ C[1 + |\ln t|] & s = 2 \\ Ct^{-(2-s)/(s-1)} & 1 < s < 2 \end{cases} \tag{A.10}$$

where in each case C is some constant (whose dependence on the model can be found from our argument), and $t = \beta_c - \beta$.

Remark. The nearest neighbor and the long-range models mentioned in the introduction are known to satisfy all the assumptions required in this lemma, with the values of s given by (A.7) and (A.8).[9,23]. Thus the above general criterion implies the bounds (1.13) and (1.14) which were stated and used in the introduction.

Proof. We start by using the observation of Sokal (Ref. 10, Appendix A) that the Gaussian domination bound and (A.3) my be combined for the useful inequality:

$$|G(p)| \leqslant \min\{(2\beta E(p))^{-1}, \chi\} \leqslant \frac{2}{2\beta E(p) + \chi^{-1}} \tag{A.11}$$

whose substitution in the Parseval identity

$$B_0 = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} dp\, |G(p)|^2 \tag{A.12}$$

yields

$$B_0 \leqslant \frac{4}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{dp}{(2\beta E(p) + \chi^{-1})^2} \tag{A.13}$$

This bound permits us to convert (A.4) (which involves both χ and B) into a bound on the derivative of an explicit function of a *single* "dynamical variable" $-\chi$. We get:

$$\begin{aligned} |J| &\leqslant \left[c_1 + \frac{4c_2(\beta\,|J|)^a}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{dp}{(2\beta E(p) + \chi^{-1})^2} \right] \left(-\frac{\partial \chi^{-1}}{\partial \beta} \right) \\ &= \frac{\partial}{\partial \beta} \left[-c_1 \chi^{-1} + \frac{4c_2(\beta\,|J|)^a}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{dp}{2\beta E(p) + \chi^{-1}} \right] \end{aligned} \tag{A.14}$$

In the second step, we used the fact that $a \geqslant 1$. An integration from β up to β_c yields

$$|J|\; t \leqslant c_1 \chi^{-1} + \frac{4c_2(\beta\,|J|)^a}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{dp}{2\beta E(p)(2\beta E(p)\,\chi + 1)} \tag{A.15}$$

The dependence of the integral in the RHS of (A.15) on the quantity χ is clearly determined by only the "density of states" $\kappa(u)$. To make this dependence explicit we use the following principle:

$$\begin{aligned} \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} F(E(p))\, dp &= \int_0^{|J|} \kappa'(u)\, F(u)\, du \\ &= F(|J|) + \int_0^{|J|} \kappa(u)\, F'(u)\, du \end{aligned} \tag{A.16}$$

for functions F such that

$$\lim_{u \to 0} F(u)\, \kappa(u) = 0 \tag{A.17}$$

For the quantity in (A.15) $F(u) = O(1/u)$ (as long as $\chi < \infty$), and the condition (A.17) is satisfied by the requirement that $s > 1$ [in (A.6)]. We get

$$|J|\, t \leqslant c_1 \chi^{-1} + 4c_2(\beta\, |J|)^a \left[\frac{1}{2\beta\, |J|\, (2\beta\, |J|\, \chi + 1)} + \int_0^{|J|} \frac{\kappa(u)}{\beta u^2 (2\beta u \chi + 1)}\, du \right] \tag{A.18}$$

With the assumption (A.6) on $\kappa(u)$, the last integral can be bounded as:

$$\int_0^{|J|} \frac{\kappa(u)}{\beta u^2(2\beta u\chi + 1)}\, du \leqslant \frac{c}{(2\beta\, |J|\, \chi)^{s-1}\, |J|} \int_0^{2\beta|J|\chi} \frac{z^{s-2}}{(z+1)}\, dz \tag{A.19}$$

If we combine this expression with the bound

$$\int_0^a \frac{z^r}{(z+1)^b}\, dz \leqslant \begin{cases} \displaystyle\int_0^\infty \frac{z^r}{(z+1)^b}\, dz & r < b-1 \\ \ln(1+a) & r = b-1 \\ \displaystyle\frac{a^{r-b+1}}{r-b+1} & r > b-1 \end{cases} \tag{A.20}$$

we obtain in (A.18):

$$t \leqslant C_1 \chi^{-1} + \begin{cases} C\chi^{-1} & s > 2 \\ C\chi^{-1} \ln(1 + 2\beta\, |J|\, \chi) & s = 2 \\ C\chi^{-(s-1)} & 1 < s < 2 \end{cases} \tag{A.21}$$

From this, the claimed inequalities (A.9) can readily be obtained.

To prove (A.10) we perform a partial integration (A.16) and use the assumption (A.6):

$$B_0 \leqslant \frac{1}{(2\beta\, |J| + \chi^{-1})^2} + 16\beta \int_0^{|J|} \frac{\kappa(u)}{(\beta u + \chi^{-1})^3}\, du \tag{A.22}$$

$$\leqslant \frac{1}{(2\beta\, |J|)^2} + \frac{2c}{(\beta\, |J|)^2\, (2\beta\, |J|\, \chi)^{s-2}} \int_0^{2\beta|J|\chi} \frac{z^s}{(z+1)^3}\, dz \tag{A.23}$$

This last inequality and (A.20) yield

$$B_0 \leqslant \frac{1}{(2\beta\,|J|)^2} + \frac{2c}{(\beta\,|J|)^2} \begin{cases} \dfrac{1}{(s-2)} & s>2 \\ \ln(1+2\beta\,|J|\,\chi) & s=2 \\ (2\beta\,|J|\,\chi)^{-(2-s)} & 1<s<2 \end{cases} \tag{A.24}$$

The bounds (A.10) follow now by combining (A.24) with (A.9). ∎

APPENDIX B. PROOF OF THE EXTRAPOLATION PRINCIPLES

In this section we discuss the derivation of the extrapolation principles presented in Sec. 2. Two principles will be proven here: one is a relation between the critical behavior along the two-phase regime $R_2 = \{\beta \geqslant \beta_c, h=0\}$ and the critical isotherm, or in fact any ray $\{t = a\beta h, h \geqslant 0\}$. The fact that this follows from the GHS inequality is an observation of Newman.[8] The other principle, which is interesting because of its agreement with the predictions of the heuristic scaling theory, establishes that if along one ray $\{t = a\beta h, h \geqslant 0\}$ the magnetization has a power law behavior (with possible logarithmic corrections), then the same behavior is asymptotically true for all other such rays.

The proof of these principles is based on the following properties of the infinite volume magnetization $M(\beta, \hat{h})$ (in this section we denote $\hat{h} = \beta h$):

(1) In the region $\hat{h} \geqslant 0$ the function $M(\beta, \hat{h})$ is continuous in $\hat{h}$. (The continuity at $\hat{h}=0$ is here just a matter of definition.)

(2) For $\hat{h} > 0$, $M(\beta, \hat{h})$ is the pointwise limit of differentiable (in fact analyric) functions M_L:

$$M = \lim_{L \to \infty} M_L(\beta, \hat{h}) \tag{B.1}$$

(3) The functions M_L satisfy the following inequalities:

$$0 \leqslant \frac{\partial M_L}{\partial \beta} \leqslant |J|\, M_L \frac{\partial M_L}{\partial \hat{h}} \tag{B.2}$$

$$0 \leqslant \frac{\partial M_L}{\partial \hat{h}} \tag{B.3}$$

The functions M_L are here the magnetizations per site on cubes of side L with the periodicized interaction $J^{(L)}_{xy} = \sum_{n \in Z^d} J_{x, y+nL}$. The functions M and M_L can be represented as the upper derivatives $M_L(\beta, \hat{h}) = -(\partial/\partial\hat{h})\, f_L(\beta, \hat{h}+0)$ of the corresponding free energy functions. By general arguments the functions $f_L(\beta, \hat{h})$ are concave in $\hat{h}$, and they converge pointwise to $f(\beta, \hat{h})$. Hence their derivatives converge for almost every $\hat{h}$. However, the systems discussed here have the additional property that the functions M_L are concave in $\hat{h}$ – for $\hat{h}>0$ (GHS inequality.) Therefore, the limiting function $M(\beta, \hat{h})$, which is continuous from above, is also concave and thus continuous for $\hat{h}>0$. This implies the pointwise convergence in (B.1) for $\hat{h}>0$. At $\hat{h}=0$, the limit $L\to\infty$ is of course discontinuous if $\beta>\beta_c$.

The inequality (B.3) and the lower bound in (B.2) are a consequence of Griffiths (II) while the upper bound in the latter follows from the GHS inequality. We remark that there is no continuity requirement in β for M; however, the previous two conditions imply that M is an increasing function of β for each $\hat{h}\geqslant 0$.

The basic observation is that (B.2) and (B.3) imply that the lines of constant magnetization

$$M_L(\beta, \hat{h}(\beta)) = m \tag{B.4}$$

have a negative slope bounded by $-|J|\, m$. Indeed, differentiating (B.4) and using (B.2), we obtain

$$0 = \frac{\partial M_L}{\partial \hat{h}} \frac{d\hat{h}}{d\beta}\bigg|_{M_L=\text{const}} + \frac{\partial M_L}{\partial \beta} \tag{B.5}$$

$$\leqslant \left(\frac{\partial M_L}{\partial \hat{h}}\right)\left(\frac{d\hat{h}}{d\beta}\bigg|_{M_L=\text{const}} + |J|\, M_L\right) \tag{B.6}$$

Therefore,

$$-|J|\, M_L \leqslant \frac{d\hat{h}}{d\beta}\bigg|_{M_L=\text{const}} \leqslant 0 \tag{B.7}$$

[The left inequality follows from (B.6) and (B.3), while the right one is a consequence of (B.5) and the lower bounds (B.2) and (B.3).]

A manifestation of this last inequality, valid even in the infinite volume limit, is provided by the following result.

Lemma B.1. Consider some $\beta_1 \geqslant 0$, $\hat{h}_1 \geqslant 0$ and let $M(\beta_1, \hat{h}_1) = m$; then

(i) $M(\beta, \hat{h}) \geqslant m$ for every $(\beta, \hat{h})$ in the region

$$\{(\beta, \hat{h}) \mid \hat{h} \geqslant \max[\hat{h}_1, \hat{h}_1 - |J|\, m(\beta-\beta_1)]\} \tag{B.8}$$

(ii) $M(\beta, \hat{h}) \leqslant m$ for every $(\beta, \hat{h})$ in the region

$$\{(\beta, \hat{h}) \mid \hat{h} \leqslant \min[\hat{h}_1, \hat{h}_1 - |J|\, m(\beta - \beta_1)]\} \tag{B.9}$$

except possibly at the corner $(\beta = \beta_1 + \hat{h}_1/|J|m, \hat{h} = 0)$.

Figure 9 presents a graphical summary of these results. In the proof of this lemma and the following results in this section we shall omit some totally elementary calculations, and present just the sketch of the arguments.

Sketch of the Proof. For finite volume and $\hat{h}_1 > 0$ the statements are an easy consequence of (B.7). The convergence properties of M_L immediately imply the bounds for the interior of the regions (B.8) and (B.9). Moreover, the continuity in $\hat{h}$ extends the bound to the boundary of those regions, except for the corner $(\beta = \beta_1 + \hat{h}_1/|J|m, \hat{h} = 0)$ where such continuity argument fails.

The special case for $\hat{h}_1 = 0$ is proven by applying the above argument to points $(\beta_1, \hat{h}_1)$ with $\hat{h}_1 \downarrow 0$ and resorting once again to the continuity in $\hat{h}$. ∎

By repeatedly applying this lemma one can prove the following two extrapolation principles.

Lemma B.2. If along a ray $t = a\hat{h}, \hat{h} \geqslant 0$

$$c_1(\hat{h})^{\alpha_1} |\ln(\hat{h})|^{\omega_1} (1 + O(\hat{h})) \leqslant M \leqslant c_2(\hat{h})^{\alpha_2} |\ln(\hat{h})|^{\omega_2} (1 + O(\hat{h})) \tag{B.10}$$

with $0 < \alpha_i < 1$ and $\omega_i \geqslant 0$, then the same inequality (with the same c_i, α_i, ω_i) is asymptotically true for any other ray $t = b\hat{h}, \hat{h} \geqslant 0$.

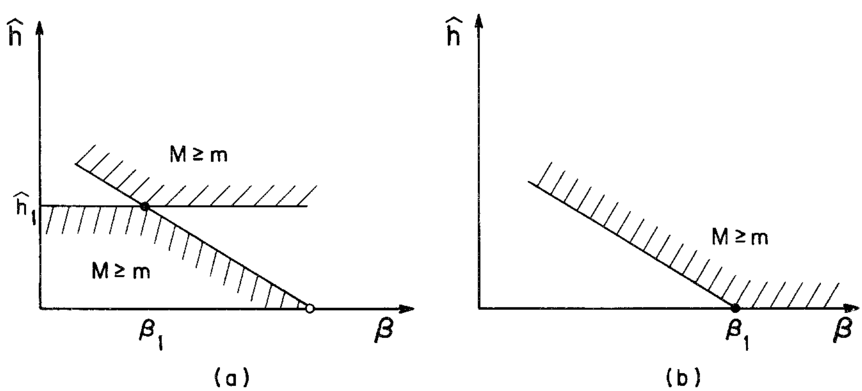


Fig. 9. Graphical summary of Lemma B.1. (a) The general situation; (b) the particular case in which $\hat{h}_1 = 0$. The slope of each slanted line is $-|J|\, m$, where m is the magnetization at the darkened point.

Lemma B.3.

(1) If along a ray $t = a\hat{h}$, $\hat{h} \geqslant 0$

$$M \leqslant c(\hat{h})^{\alpha} \, |\ln(\hat{h})|^{\omega} \, (1 + O(\hat{h})) \tag{B.11}$$

with $0 < \alpha < 1$ and $w \geqslant 0$, then in the region $R_2 = \{(\beta, \hat{h} = 0) \mid \beta > \beta_c\}$

$$M \leqslant (|J| \, c^{1/\alpha})^{\alpha/(1-\alpha)} \, |t|^{\alpha/(1-\alpha)} \, |\ln(|J| M \, |t|)|^{\omega/(1-\alpha)} (1 + O(|t|) \tag{B.12}$$

(2) If in region R_2

$$M \geqslant c \, |t|^{\lambda} \, (1 + O(t)) \tag{B.13}$$

with $\lambda \geqslant 0$; then, along any ray $t = a\hat{h}$, $\hat{h} \geqslant 0$,

$$M \geqslant (|J| c^{1/\lambda})^{\lambda/(1+\lambda)} \, (\hat{h})^{\lambda/(1+\lambda)} (1 + O(\hat{h})) \tag{B.14}$$

Sketch of the Proof of Lemma B.2. Consider two rays with slopes a_1 and a_2 respectively. Lemma B.1 implies the situation summarized in Fig. 10, which shows that every point of one of the rays has the magnetization bounded above and below by the magnetization at points of the other ray. For instance,

$$M(P_0) \leqslant M(P_1) \leqslant M(P_2)$$

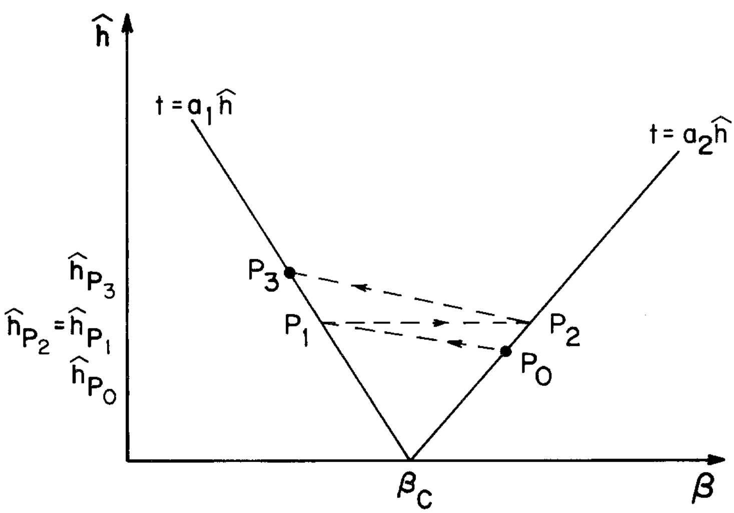


Fig. 10. Summary of the argument for the proof of Lemma B.2. The arrows indicate the direction of increasing magnetization. The slope of each tilted dashed line is $-|J|$ times the magnetization at the corresponding darkened point. Under the assumption of Lemma B.2 these slopes are vanishingly small for P_0, P_3 close to $(\beta_c, \hat{h} = 0)$.

Moreover, $\hat{h}_{P_1} = \hat{h}_{P_2}$ while P_1 and P_0 are connected by a straight line of slope not larger (in absolute value) than $|J|\, M(P_0)$. Hence, if the magnetization obeys a power law (upper) bound in the ray of slope a_2, the difference among $\hat{h}_{P_1}$, $\hat{h}_{P_0}$, and $\hat{h}_{P_2}$ becomes negligible for points of the ray close to the critical point ($t = 0$, $\hat{h} = 0$). Hence, asymptotically the same power law is obeyed by the points of the ray of slope a_1. ∎

The proof of Lemma B.3 is based on similar considerations.

APPENDIX C. DIFFERENTIAL INEQUALITIES FOR SPINS WITH MEASURES IN THE GRIFFITHS–SIMON CLASS

In this appendix we present an extension of one of the inequalities derived in this paper to a more general class of spin measures—the Griffiths–Simon class described below. At the outset, let us point out that we do not have such a full generalization of Theorem 5.7 which is the basis for the results derived in Sec. 1 for Ising systems. The result discussed here is the following more general version of the inequality (5.19).

Theorem C.1. For any finite ferromagnetic system in $[-L, L]^d$ with periodic interactions and spin measures in the Griffiths–Simon class, we have

$$\frac{\partial M}{\partial \beta} \geqslant \frac{|M|\, J\, |\chi - (\frac{1}{4} + (\beta\, |J|)^2\, B_0)\, h\chi|_+}{\frac{3}{2} + 2(\beta\, |J|)^2\, B_0} \tag{C.1}$$

Before discussing the proof let us briefly describe the Griffiths–Simon class of spin measures. Its definition is based on the observation of Griffiths[24] that a number of interesting spin distributions can be generated by representating the spin variable as a weighted sum of ferromagnetically coupled Ising spins. By taking limits of such variables, Simon and Griffiths[24] further enlarged the class of spin measures which inherit some of the properties of Ising spins and showed that it contains the "ϕ^4" variables [with $\rho_0(d\phi) = \exp(-\lambda\phi^4 + b\phi^2)\, d\phi$].

For a convenient representation of a system of G.S. spins, one may regard each lattice site as a "block" of "microscopic" sites, with an internal index $\alpha = 1,..., N$, and each spin variable ϕ as a weighted average of "microscopic" Ising spins $\sigma_{(x,\alpha)}$:

$$\phi_x = \sum_{\alpha=1}^{N} Q_\alpha^{(N)} \sigma_{(x,\alpha)} \tag{C.2}$$

with positive coefficients $Q_\alpha^{(N)}$.

The Griffiths–Simon class of measures is the set of measures $\rho_0(\phi)\,d\phi$ induced for ϕ by letting the microscopic Ising spins be either independent or coupled with a ferromagnetic intrablock Hamiltonian of the form

$$H_1 = -\frac{1}{2}\sum_x \sum_{\alpha,\beta} I^{(N)}_{x;\alpha,\beta}\,\sigma_{(x,\alpha)}\sigma_{(y,\beta)} \tag{C.3}$$

and taking limits of such measures with $N \to \infty$. To ensure convergence as $N \to \infty$ of the finite volume correlation function, we shall always assume that $\int e^{a\phi^2}\rho_0(d\phi) < \infty$ for all $a < \infty$. This class of measures includes discrete as well as continuous spin variables and, in particular, the ϕ^4 measure mentioned above (for which $Q^{(N)}_\alpha$ is independent of α).

The Griffiths–Simon representation of some continuous spin systems as limits of Ising spins provides a perspective which is complementary to the standard interpretation of Ising spins as the strong coupling limit of ϕ^4 continuous variables. Furthermore, from the point of view of this work this representation is a convenient tool for the extension of results proved for the Ising models via random current representations, to a broader family of models. Indeed, a model with the two-body interaction

$$H_2 = -\frac{1}{2}\sum_{x,y} J_{x,y}\phi_x\phi_y - \sum_x h_x\phi_x \tag{C.4}$$

and spins in the G.S. class can be approximated by Ising models with a total Hamiltonian

$$H = H_1 + H_2 = -\frac{1}{2}\sum_{\underline{x},\underline{y}} J_{\underline{x}\underline{y}}\sigma_{\underline{x}}\sigma_{\underline{y}} - \sum_{\underline{x}} h_{\underline{x}}\sigma_{\underline{x}} \tag{C.5}$$

for which we can apply the random current representation. We have denoted here $\underline{x} = (x, \alpha)$; $\underline{y} = (y, \beta)$ and:

$$J_{\underline{x}\underline{y}} = \begin{cases} J_{xy}Q^{(N)}_\alpha Q^{(N)}_\beta & \text{if} \quad y \neq x \\ I^{(N)}_{x;\alpha,\beta} & \text{if} \quad y = x \end{cases} \tag{C.6}$$

$$h_{\underline{x}} = h_x Q^{(N)}_\alpha$$

To obtain results applicable to the whole Griffiths–Simon class, we must maneuver so as to arrive at expressions which are stated only in terms of the block spins $\{\phi_x\}$ and the interblock couplings J_{xy}, with no reference to the microscopic variables σ or any intrablock parameter. Such expressions are insensitive to the limit $N \to \infty$ and are automatically valid for all measures in the Griffiths–Simon class regardless of the particular forms of $Q^{(N)}$ and $I^{(N)}$. To comply with this program we shall try, as a general policy, to replace connection constraints involving sites by constraints referring to blocks. For this purpose, the following result provides an useful extension of the principle derived in Lemma 3.1. For each $z \in \Lambda$ let us denote $B_z = \{(z, \alpha) \mid 1 \leqslant \alpha \leqslant N\}$($z$-block.)

Proposition C.2. Let f be any positive function on current configurations. For any $z \neq x$:

$$\sum_{\substack{\partial \underline{n}_1 = \{\underline{x}\} \Delta \{\underline{y}\} \\ \partial \underline{n}_2 = \varnothing}} W(\underline{n}_1)\, W(\underline{n}_2)\, f(\underline{n}_1 + \underline{n}_2)\, \mathrm{I}\,[\underline{n}_1 + \underline{n}_2 \colon \underline{x} \to B_z]$$

$$\leqslant \sum_{\substack{\delta \\ \underline{p}\colon p \neq z}} \beta J_{\underline{p},(z,\delta)} \sum_{\substack{\partial \underline{n}_1 = \{\underline{x}\} \Delta \{\underline{p}\} \\ \partial \underline{n}_2 = \{(z,\delta)\} \Delta \{\underline{y}\}}} W(\underline{n}_1)\, W(\underline{n}_2)\, f(\underline{n}_1 + \underline{n}_2) \tag{C.7}$$

The proof of this proposition can be found in Ref. 3 (Proposition 7.1). We now turn to the proof of the inequality (C.1).

Proof of Theorem C.1. The proof follows the same basic guidelines as that of Theorem 5.6 except that some conditions related with sites are replaced by conditions over blocks. For $M = \langle \phi_0 \rangle = \sum_\alpha Q_\alpha \langle \sigma_{(0,\alpha)} \rangle$ we have an expression analogous to (5.21) but involving the coupling constants of the Hamiltonian (C.5):

$$\frac{\partial M}{\partial \beta} = \frac{1}{2} \sum_{\substack{\underline{u},\underline{v} \\ \alpha}} J_{\underline{u}\underline{v}} Q_\alpha \left\{ \left[\sum_{\substack{\partial \underline{n}_1 = \{(0,\alpha)\} \Delta \{\underline{u}\} \\ \partial \underline{n}_2 = \varnothing}} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \right.\right.$$

$$\left.\left. \times \mathrm{I}\,[\underline{n}_1 + \underline{n}_2 \colon (0, \alpha) \nrightarrow h] \langle \sigma_{\underline{v}} \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(0,\alpha)} \right] + [\underline{u} \Leftrightarrow \underline{v}] \right\} \tag{C.8}$$

At this point we substitute the cluster $C_{\underline{n}_1 + \underline{n}_2}(0, \alpha)$, whose bonds end in a *site* connected to $\underline{0}$, by the larger cluster

$$\bar{C}_{\underline{n}}(\underline{0}) = \{\{\underline{y}\, \underline{z}\} \mid \underline{n} \colon \underline{0} \to B_y \text{ or } \underline{n} \colon \underline{0} \to B_z\} \cup \{\{\underline{y}\, h_y\} \mid \underline{n} \colon \underline{0} \to B_y\} \tag{C.9}$$

formed by bonds ending in a *block* connected to $\underline{0}$. By Griffiths (II):

$$\langle \sigma_{\underline{v}} \rangle_{C^c_{\underline{n}_1 + \underline{n}}(0,\alpha)} \geqslant \langle \sigma_{\underline{v}} \rangle_{\bar{C}^c_{\underline{n}_1 + \underline{n}_2}(0,\alpha)} \tag{C.10}$$

We now use (C.10) in the RHS of (C.8) and then Lemma 5.5 to obtain, as in (5.23),

$$\frac{\partial M}{\partial \beta} \geqslant M\, |J|\, \chi - \mathrm{II} - \mathrm{III} \tag{C.11}$$

with

$$\mathrm{II} = \sum_{\substack{u,v,k\\ \alpha}} Q_\alpha \tanh(\beta h_k)\, K(v, k)\, J_{uv} \sum_{\substack{\partial n_1 = \{(0,\alpha)\}\Delta\{u\}\\ \partial n_2 = \varnothing}} \frac{W(n_1)}{Z}\frac{W(n_2)}{Z}$$
$$\times \mathrm{I}\,[n_1 + n_2 \colon (0,\alpha) \nrightarrow h]\; \mathrm{I}\,[n_1 + n_2 \colon (0,\alpha) \to B_k] \tag{C.12}$$

and

$$\mathrm{III} \leqslant \sum_{\substack{u,v,k,l\\ \alpha}} Q_\alpha K(v, k)\, J_{uv} \beta J_{kl} \sum_{\substack{\partial n_1 = \{(0,\alpha)\}\Delta\{u\}\\ \partial n_2 = \varnothing}} \frac{W(n_1)}{Z}\frac{W(n_2)}{Z}$$
$$\times \mathrm{I}\,[n_1 + n_2 \colon (0,\alpha) \nrightarrow h]\; \mathrm{I}\,[n_1 + n_2 \colon (0,\alpha) \to B_k] \langle \sigma_l \rangle_{C^c_{n_1+n_2}(0,\alpha)} \tag{C.13}$$

Let us work with II. We must separate the terms for $k \neq 0$, for which we use (C.7), and the terms for $k = 0$.

$$\mathrm{II} = \mathrm{II}(k \neq 0) + \mathrm{II}(k = 0)$$

$$\mathrm{II}(k \neq 0) \leqslant \sum_{\substack{u,v\\ k\colon k \neq 0\\ \alpha}} Q_\alpha \tanh(\beta h_k)\, K(v, k)\, J_{u,v} \sum_{\substack{w\colon w \neq k\\ \delta}} \beta J_{w,(k,\delta)}$$
$$\times \sum_{\substack{\partial n_1 = \{(0,\alpha)\}\Delta\{w\}\\ \partial n_2 = \{u\}\Delta\{(k,\delta)\}}} \frac{W(n_1)}{Z}\frac{W(n_2)}{Z}\, \mathrm{I}\,[n_1 + n_2 \colon (0,\alpha) \nrightarrow h]$$
$$\leqslant \sum_{\substack{u,v\\ k\colon k \neq 0\\ \alpha}} \beta h_k \langle \sigma_v \sigma_k \rangle_{h=0} J_{u,v} \sum_{\substack{w\colon w \neq k\\ \delta}} \beta J_{w,(k,\delta)}$$
$$= \sum_{\substack{u,v\\ k \neq 0\\ w \neq k}} \beta h J_{uv} \langle \phi_v \phi_k \rangle_{h=0} \beta J_{wk} \langle \phi_u \phi_k \rangle_{h=0} \langle \phi_0, \phi_w \rangle$$

In the inequality we used (4.22), Griffiths (II) as explained immediately after (5.25), and the fact that $\tanh x \leqslant x$. We relax the restrictions on k and w and use Schwartz inequality (5.26) and translation invariance to obtain

$$\mathrm{II}(k \neq 0) \leqslant (\beta\, |J|)^2\, B_0 h \chi \tag{C.14}$$

On the other hand, the part of the sum in (C.12) with $k = 0$ is:

$$\mathrm{II}(k = 0) = \sum_{\substack{u,v\\ \alpha,\delta}} Q_\alpha \tanh(\beta h_{(0,\alpha)})\, K(v, (0,\alpha))\, J_{uv} \langle \sigma_{(0,\alpha)}, \sigma_u \rangle$$
$$\leqslant \sum_{u,v} \beta h \langle \phi_v \phi_0 \rangle_{h=0} J_{uv} \langle \phi_0, \phi_u \rangle$$

The inequality follows from (4.22) and the bound $\tanh x \leqslant x$. Using Schwartz inequality and translation invariance we obtain

$$\mathrm{II}(k=0) \leqslant \beta\, |J|\, B_0^{1/2} h\chi \tag{C.15}$$

(C.14) and (C.15) can be combined, for instance, by resorting to the elementary inequality $x \leqslant x^2 + 1/4$. The result is:

$$\mathrm{II} \leqslant [(\beta\, |J|)^2\, B_0 + \tfrac{1}{4}]\, h\chi \tag{C.16}$$

The work with III is very similar, except that we first replace $\bar{C}$ by C in the RHS of (C.13), which gives an upper bound by virtue of (C.10). We use (C.7) for the terms with $k \neq 0$ and operate with the resulting expressions in a way almost identical to the one used for the Ising Model. The result is:

$$\mathrm{III} \leqslant 2[(\beta\, |J|)^2\, B_0 + \tfrac{1}{4}]\, \frac{\partial M}{\partial \beta} \tag{C.17}$$

The proposed bound (C.1) follows by the substitution of (C.16) and (C.17) in (C.11). ∎

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